Mohammed K A Kaabar — Telescoping & Geometric Series — October 15th, 2015

* Infinite Series:

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots$$

Example 1 Find the 1st five partial sums for $1+2+3+4+5+\cdots$

Solution: $S_1 = 1$

$S_2 = 1+2 = 3$

$S_3 = 1+2+3 = 6$

$S_4 = 1+2+3+4 = 10$

$S_5 = 1+2+3+4+5 = 15$

$$* \boxed{\sum_{k=1}^{\infty} a_k} = \lim_{n\to\infty} \boxed{\sum_{k=1}^{n} a_k} = \lim_{n\to\infty} \boxed{S_n}$$

Example 2 Find the sum for the sequence: $\sum_{k=2}^{\infty} \ln\left(\frac{k}{k+1}\right)$.

final answer

Solution: $$\sum_{k=2}^{\infty} \ln\left(\frac{k}{k+1}\right) = \lim_{n\to\infty} \boxed{\sum_{k=2}^{n} \ln\left(\frac{k}{k+1}\right)} = \lim_{n\to\infty} [\ln(2) - \ln(n+1)] = \boxed{-\infty}$$ diverges

We know that $$\sum_{k=2}^{n} \ln\left(\frac{k}{k+1}\right) = \sum_{k=2}^{n} [\ln(k) - \ln(k+1)] =$$

$$= [\ln(2) - \cancel{\ln(3)}] + [\cancel{\ln(3)} - \cancel{\ln(4)}] + [\cancel{\ln(4)} - \cancel{\ln(5)}] + [\cancel{\ln(5)} - \cancel{\ln(6)}]$$

$$+ \cdots + [\cancel{\ln(n-1)} - \cancel{\ln(n)}] + [\cancel{\ln(n)} - \ln(n+1)] = \boxed{\ln(2) - \ln(n+1)}$$

this is called telescoping series. ☑

Example ③: Find the sum for the following sequence:

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)}.$$

Solution: $$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \lim_{n\to\infty} \boxed{\sum_{k=1}^{n} \frac{1}{k(k+1)}}$$

Now, let's use partial fractions as follows:

$$\frac{1}{k(k+1)} = \frac{A}{k} + \frac{B}{k+1}$$ "Cover Method"

$k=0$ $\quad$ $k=-1$

$A = \frac{1}{0+1} = 1 \Rightarrow \boxed{A=1}$

$B = \frac{1}{-1} = -1 \Rightarrow \boxed{B=-1}$

So, $\boxed{\frac{1}{k(k+1)} = \frac{1}{k} + \frac{-1}{k+1}}$

Hence, $$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left[\frac{1}{k} - \frac{1}{k+1}\right] = \left[1 - \cancel{\frac{1}{2}}\right] + \left[\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right] +$$

$$\left[\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right] + \left[\cancel{\frac{1}{4}} - \cancel{\frac{1}{5}}\right] + \cdots + \left[\cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n}}\right] + \left[\cancel{\frac{1}{n}} - \frac{1}{n+1}\right] =$$

$$= \boxed{1 - \frac{1}{n+1}}$$

Thus, $$\lim_{n\to\infty} \left[1 - \frac{1}{n+1}\right] = 1 - \cancel{\frac{1}{\infty}}^{\,0} = \boxed{1}$$ Converges. ☒

* Geometric Series

$$a+ar+ar^2+ar^3+\cdots=\sum_{\boxed{n=0}}^{\infty} a\boxed{r^n}=\sum_{n=1}^{\infty} ar^{n-1}$$

Example 4: Find the sum of the sequence: $\sum_{n=1}^{\infty} 2^{2n}3^{1-n}$.

Solution: $\sum_{n=1}^{\infty} 2^{2n}\cdot 3^{1-n}=\sum_{n=1}^{\infty} 4^n\cdot 3^1\cdot 3^{-n}$

$$=\sum_{n=1}^{\infty}\frac{4^n\cdot 3^1}{3^n}=\sum_{n=1}^{\infty}3\left(\frac{4}{3}\right)^n=\boxed{4+\frac{16}{3}+\cdots}$$

$\boxed{r=\frac{4}{3}\approx 1.33>1}$ $\boxed{\text{diverges}}$ ☒

Example 5: Find the sum of the sequence: $\sum_{n=0}^{\infty}\frac{\pi^n}{3^{n+1}}$.

Solution: $\sum_{n=0}^{\infty}\frac{\pi^n}{3^{n+1}}=\sum_{n=0}^{\infty}\frac{\pi^n}{3^n\cdot 3^1}=\sum_{n=0}^{\infty}\frac{1}{3}\left(\frac{\pi}{3}\right)^n=\boxed{\frac{1}{3}+\frac{\pi}{9}+\cdots}$

$\boxed{r=\frac{\pi}{3}\approx 1.0471>1}$ $\boxed{\text{diverges}}$

Example 6: Find the sum of the sequence: $\sum_{n=0}^{\infty}\frac{1}{(\sqrt{2})^n}$.

Solution: $\sum_{n=0}^{\infty}\frac{1}{(\sqrt{2})^n}=\sum_{n=0}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^n=\frac{1}{1-\frac{1}{\sqrt{2}}}=\boxed{2+\sqrt{2}\approx 3.4142}$.

$$\boxed{\begin{array}{l} r=\frac{1}{\sqrt{2}}\approx 0.707<1 \text{ converges}\\ a=1\end{array}}$$

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