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Section: -Solution-

Activity: Differential Activity

Directions and Overview

This activity will introduce differential equations, describe a model of infusion of drugs into the bloodstream, and introduce the solution of differential equations. Please do the numbered activities within the text below. Try to be neat and organized. Justify your reasoning, and remember to show your work.

Introduction

A **Differential** Equation, or DE, is an equation involving derivatives of a function. An Ordinary Differential equation, or ODE, has only one independent variable.

Some generic examples of ODEs are given below:

$y' = ky$ linear Differential Equation. (1)

$y' = ky + b$ Linear Differential Equation. (2)

$y" = ky$ Linear Differential Equation. (3)

$y" + ty' = 1$ linear Differential Equation. (4)

$y' = ky(M - y) = KyM - Ky^2$ Since y^2 is NOT to the power 1, then it's Non-linear Differential Equation. (5)

In these examples $y' = \frac{dy}{dt}$, $y" = \frac{d^2y}{dt^2}$, and $y(t)$ is the function with independent variable t. In a typical problem the function y is initially unknown, so it is sometimes referred to as the 'unknown function'. Of course, different symbols could be used for the independent variable and the unknown function. The 'order' of a differential equation is the order of the highest derivative in the differential equation.

Recall that a linear function of x has the form $mx + b$, where m and b do not depend on x. A linear function of two variables x and y has the form $ax + by + c$ where a, b and c do not depend on x or y. A general 'linear' first order ODE has the form $ay + by' + c = 0$, where a, b and c do not depend on y or y' but may depend on t. An ODE that is not linear is said to be 'nonlinear'.

I_1: For each of the examples $1 \to 5$ given above, determine the order of the ODE and whether it is linear or nonlinear, and why. Assume that k, b and M do not depend on y.
Solution:

① Order 1, linear, the dependent variable and all its derivatives are to the power 1.

② Order 1, linear, " " " " " " " ".

③ Order 2, linear, " " " " " " " ".

③ Order 2, linear, " " " " " " " ".

④ Order 1, Non-linear, because the dependent variable (y^2) is NOT to the power 1.

1. Exponential decay/growth is modeled by the equation $y(t) = y_0 e^{kt}$ for $t \geq 0$, y_0 the initial amount of substance, and k the decay/growth rate (a constant). Observe that if y represents the amount of substance after t time then $y' = \frac{dy}{dt}$ represents the rate of change of amount of substance.
Show, by differentiating the formula for $y(t)$ that the resulting equation is an ODE, identify which generic form $(1, 2, ..., 5)$ it takes on.

$$y'(t) = k\boxed{y_0 e^{kt}} = k\,y(t)$$

2. A 'separable' first-order ODE is an equation that can be written in the form:

$$g(y)y'(t) = h(t) \tag{6}$$

where g and h are continuous functions. We try to solve these equations for the unknown function y by integrating both sides with respect to t as follows:

$$\int (g(y)y'(t))\,dt = \int h(t)dt$$

Notice that from calculus 1 we know that if y is a function of x then $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = y'(x)$. This implies that $dy = y'(x)dx$. This means for our integral we have $y'(t)dt = dy$, giving us

$$\int g(y)dy = \int h(t)dt$$

(a) Use this method to solve the general ODE

$$y'(t) = ky$$

Remember: $y'(t) = \frac{dy}{dt}$

So, $\frac{dy}{dt} = ky \Rightarrow \frac{dy}{dt} \times \frac{k}{y^{-1}} \Longrightarrow y^{-1}dy = kdt \Rightarrow y^{-1}dy - kdt = 0$

$\Rightarrow \int \frac{1}{y}dy - \int kdt = \int 0 \Rightarrow \ln|y| - kt = 0 + C \Rightarrow \boxed{\ln|y| - kt = C}$

We can continue solving $\ln|y| - kt = C$ as follows: $\boxed{\text{Assume } C = e^C}$

$\ln|y| = kt + C \Rightarrow e^{\ln|y|} = e^{kt+C} \Rightarrow e^{\ln|y|} = e^{kt} \cdot \boxed{e^C}$

$\Rightarrow y = e^{kt} \cdot C \Rightarrow \boxed{y = Ce^{kt}}$ where C is a constant.

$\Rightarrow$ Therefore, $\boxed{y = Ce^{kt}}$ where C is a constant.

(b) If we assume the solution is a exponential growth equation, what is the meaning of y_0 in application, and what does the assumption $y > 0$ tell you about y_0?

$y(t) = y_0 e^{kt}$, $y(0) = y_0 \equiv$ Initial Condition.

3. Suppose an intravenous line provides a continuous flow of a drug directly into your bloodstream at rate b. Your liver or kidneys filter the drug out of your blood at a rate ky that is proportional to the amount of drug y in your bloodstream. The amount of drug y in your body is influenced by both of these processes. A differential equation modeling the amount of drug is given by example (2). Solve (2) using the separation of variables technique under the following assumptions:

(a) The infusion begins at time $t = 0$ at which point there is no drug in your body.

(b) We know that $y < \frac{b}{k}$ for all $t \geq 0$.

$\frac{dy}{dt} = b - ky \Rightarrow \frac{dy}{dt} = \frac{b}{1} - \frac{k}{y^{-1}} \Rightarrow \frac{dy}{dt} \times \frac{by^{-1} - k}{y^{-1}} \Rightarrow y^{-1}dy = (by^{-1} - k)dt$

Now, divide everything by $(by^{-1} - k)$, we obtain: $\frac{y^{-1}}{by^{-1} - k} dy = dt$

$\Rightarrow \frac{1}{y}\left(\frac{1}{by^{-1} - k}\right)dy = dt \Rightarrow \frac{1}{b - ky}dy = dt \Rightarrow \frac{1}{b - ky}dy - dt = 0$

$\Rightarrow \int \frac{1}{b - ky} dy - \int dt = \int 0 \Rightarrow \int \frac{1}{u} \cdot \frac{du}{k} - t = 0 + C$

$\Rightarrow \frac{-1}{k}\int \frac{1}{u} du - t = C \Rightarrow \frac{-1}{k} \ln|u| - t = C$

$\boxed{\text{by substitution } u = b - ky \quad du = -k\,dy \Rightarrow dy = \frac{-du}{k}}$

$\Rightarrow \frac{-1}{k} \ln|b - ky| - t = C$ Now, multiply both sides

by $-k$. Note kC assume it's equivalent to C. $\Rightarrow \ln|b - ky| + kt = C$

$\boxed{\text{assume it } e^C = C}$

$\ln|b - ky| = C + kt \Rightarrow e^{\ln|b-ky|} = e^{-kt + C} \Rightarrow b - ky = e^{-kt} e^C$

$\Rightarrow b - ky = Ce^{-kt} \Rightarrow ky = b - Ce^{-kt} \Rightarrow \boxed{y = \frac{b}{k} - \frac{Ce^{-kt}}{k}}$ $\boxed{\text{Assume } C = \frac{C}{k}}$ $\Rightarrow \boxed{y(t) = \frac{b}{k} - Ce^{-kt}}$

at $y(0) = \frac{b}{k} - Ce^{-k(0)} = 0 \Rightarrow \frac{b}{k} - C = 0 \Rightarrow \frac{b}{k} = C \Rightarrow \boxed{C = \frac{b}{k}}$

Hence, $y(t) = \frac{b}{k} - \frac{b}{k}e^{-kt} = \boxed{\frac{b}{k}(1 - e^{-kt})}$

4. Suppose an antibiotic with half-life $\boxed{T_{\frac{1}{2}} = 12hr}$ is given to a patient intravenously at a rate of $\boxed{b = 50mg/hr}$.

(a) Find the rate constant k. From problem 3, we have: $\boxed{y(t) = \frac{b}{k}(1 - e^{-kt})}$

From the mathematical biology class, we learned that the half-life equation is: $\boxed{T_{1/2} = \frac{\ln 2}{k} = \text{constant}}$. So, from this equation, we can find k easily as follows: $12 = \frac{\ln(2)}{k} \Rightarrow k = \frac{\ln(2)}{12} \approx \boxed{0.0578}$

(b) Use a program to graph the drug function y for $0 \le t \le 48hr$.

TA Initials: MK (USE MAPLE SOFTWARE)

So, $\boxed{y(t) = \frac{50}{0.0578}(1 - e^{-0.0578t})}$ ⇐ This is the required function.

(c) A 'steady-state' or 'equilibrium' solution of a differential equation is a solution that does not change with time. For example, a steady state solution of $\frac{dy}{dt} = f(t)$ is a solution that satisfies that equation with $f(t) = 0$.

i. What is the steady state solution for the amount of drug delivered by infusion in problem ~~5~~ 3?

From problem 3, $\frac{dy}{dt} = b - ky \Rightarrow \frac{dy}{dt} = b - ky = 0 \Rightarrow b - ky = 0$

$$\Rightarrow ky = b \Rightarrow \boxed{y = \frac{b}{k}}$$

Hence, the steady-state solution is $\boxed{y = \frac{b}{k}} \Rightarrow \boxed{y(t) = \frac{b}{k}}$

ii. Find $\lim_{t\to\infty} y(t)$

$$\Rightarrow \lim_{t\to\infty} y(t) = \lim_{t\to\infty} \frac{b}{k} = \boxed{\frac{b}{k}}$$

iii. Verify that your results seem consistent with the graph that you generated in problem ~~5~~ 3. What is the steady state of a drug delivered by infusion in terms of b and k.

Verify that from your graphs (Depending on students' graphs).