Ex 1 | Find $\lim_{x\to\infty} \frac{1}{x} = ?$

Solution:

$$\lim_{x\to\infty} \frac{1}{x} = \frac{1}{\infty} = 0$$

* Important Notes

① $\frac{1}{\pm\infty} = 0$

② $\infty + \infty = \infty$

③ $\infty - \infty = \boxed{\text{indeterminate}}$

④ $\infty \cdot \infty = \infty$

⑤ $\frac{\infty}{\infty} = \boxed{\text{indeterminate}}$

Ex 2 | Find $\lim_{x\to+\infty} \frac{x^2 - x + 1}{2x^2+3} = ?$

Solution: We can find it in three different ways:

I. If the degree of numerator = the degree of denomenator, then the answer $= \boxed{\frac{1}{2}}$

II. If we use the leading terms, we obtain:

$$\lim_{x\to\infty} \frac{\cancel{x^2}}{2\cancel{x^2}} = \boxed{\frac{1}{2}}$$

III. If we divide by the highest exponent in denomenator, we obtain:

$$\lim_{x\to\infty} \frac{\frac{x^2}{x^2} - \frac{x}{x^2} + \frac{1}{x^2}}{\frac{2x^2}{x^2} + \frac{3}{x^2}} = \lim_{x\to\infty} \frac{1 - \cancelto{0}{\frac{1}{x}} + \cancelto{0}{\frac{1}{x^2}}}{2 + \cancelto{0}{\frac{3}{x^2}}} = \boxed{\frac{1}{2}}$$

**Ex3** Find $\lim_{x\to-\infty} \frac{2x^2-x+1}{5x^3+x^2-2} = ?$.

Solution: We can find it in three different ways:

I. If the degree of numerator < the degree of denomenator, then the answer is $\boxed{\text{zero}}$.

II. If we use leading terms, we obtain:

$$\lim_{x\to-\infty} \frac{2\cancel{x^2}}{5x^{\cancel{3}}} = \lim_{x\to-\infty} \frac{2}{5x} = \frac{2}{-\infty} = \boxed{0}$$

III. If we divide by the highest exponent in denomenator, we obtain:

$$\lim_{x\to-\infty} \frac{\frac{2x^2}{x^3}-\frac{x}{x^3}+\frac{1}{x^3}}{\frac{5x^3}{x^3}+\frac{x^2}{x^3}-\frac{2}{x^3}} = \lim_{x\to-\infty} \frac{\cancel{\frac{2}{x}}^{\nearrow 0}-\cancel{\frac{1}{x^2}}^{\nearrow 0}+\cancel{\frac{1}{x^3}}^{\nearrow 0}}{5+\cancel{\frac{1}{x}}^{\nearrow 0}-\cancel{\frac{2}{x^3}}^{\nearrow 0}} = \frac{0}{5} = \boxed{0}$$

**Ex4** Find $\lim_{x\to+\infty} \frac{-2x^4+x^2-1}{x^2-x+3} = ?$

Solution: We can find it in three different ways:

I. If the degree of numerator > the degree of denomenator then, the answer is $\boxed{\pm\infty}$

II. If we use leading terms, we obtain:

$$\lim_{x\to-\infty} \frac{-2x^{\cancel{4}\,2}}{\cancel{x^2}} = \lim_{x\to-\infty} -2x^2 = \boxed{-\infty}$$

III. If we divide by the highest exponent in denomenator, we obtain:

$$\lim_{x\to-\infty} \frac{\frac{-2x^4}{x^2}+\frac{x^2}{x^2}-\frac{1}{x^2}}{\frac{x^2}{x^2}-\frac{x}{x^2}+\frac{3}{x^2}} = \lim_{x\to\mp\infty} \frac{-2x^2+1-\cancel{\frac{1}{x^2}}^{\nearrow 0}}{1-\cancel{\frac{1}{x}}^{\nearrow 0}+\cancel{\frac{3}{x^2}}^{\nearrow 0}} = \boxed{-\infty}$$

## * Continuity:

**Definition:** We say that a function, $f(x)$, is continuous at $x=a$ if it satisfies the following:

① $f(a)$ exists

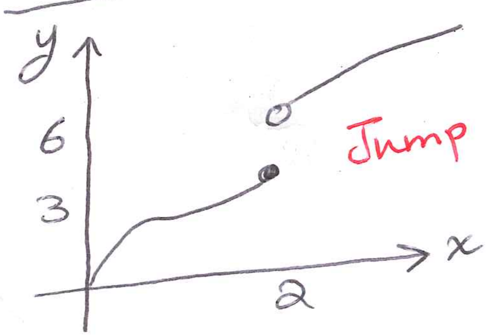
② $\lim\limits_{x\to a} f(x)$ exists

③ $\lim\limits_{x\to a} f(x) = f(a)$

Otherwise, we say $f$ is discontinuous at $a$.

**Ex5** Discuss the continuity for each of the following:

**Part a:**




**Solution:** ① $f(2) = 3$

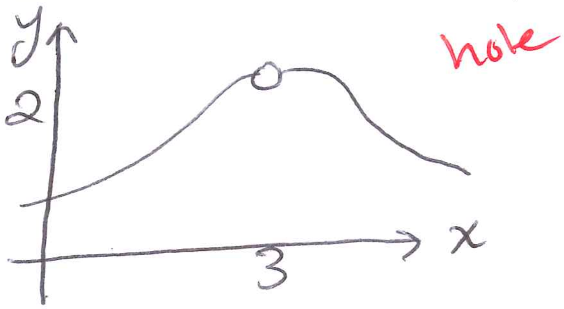
② $\lim\limits_{x\to 2^+} f(x) = 6$

③ $\lim\limits_{x\to 2^-} f(x) = 3$

$\Big\} \Rightarrow \lim\limits_{x\to 2} f(x) = \text{DNE}$

So, $\lim\limits_{x\to 2} f(x) \neq f(2) \Rightarrow$ It's discontinuous at $x=2$ due to Jump.

Part b:




Solution:

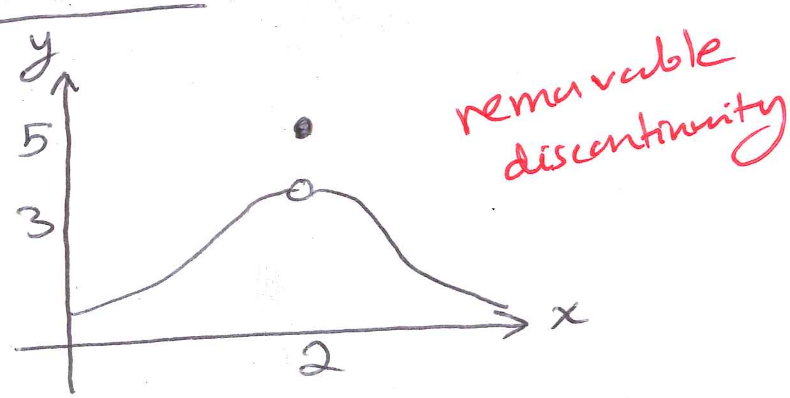
① $f(3) = DNE$

② $\lim_{x \to 3^+} f(x) = 2$

③ $\lim_{x \to 3^-} f(x) = 2$

So, $\lim_{x \to 3} f(x) = 2 \Rightarrow$ Thus, $\lim_{x \to 3} f(x) \neq f(3)$ which implies that $f$ is discontinuous at 3 due to hole.

Part c:




Solution:

① $f(2) = 5$

② $\lim_{x \to 2^+} f(x) = 3$

③ $\lim_{x \to 2^-} f(x) = 3$

$\Rightarrow \lim_{x \to 2} f(x) = 3$

④ $\lim_{x \to 2} f(x) \neq f(2) \Rightarrow$ Thus, $f$ is discontinuous at 2 due to removable discontinuity.

***theorem 1:** The following functions are continuous whenever they are defined:

1. Polynomials.
2. Rational Functions
3. Trig. Functions
4. Roots
5. ln & Exponential Functions

**Ex6** Discuss the continuity for each of the following:

**Part a:** $f(x) = x^7 - x^3 + 11$

**Solution:** Continuous on $(-\infty, \infty)$ "everywhere" because it's a polynomial function.

**Part b:** $f(x) = \frac{x}{9-x^2}$

**Solution:** Continuous where $x \neq \pm 3$ because $9 - x^2 = 0$, $x^2 = 9$, $x = \pm 3$

**Part c:** $f(x) = \sqrt[3]{x-2}$

**Solution:** Cubic root is always continuous everywhere which means that $f(x)$ is continuous on $(-\infty, \infty)$.

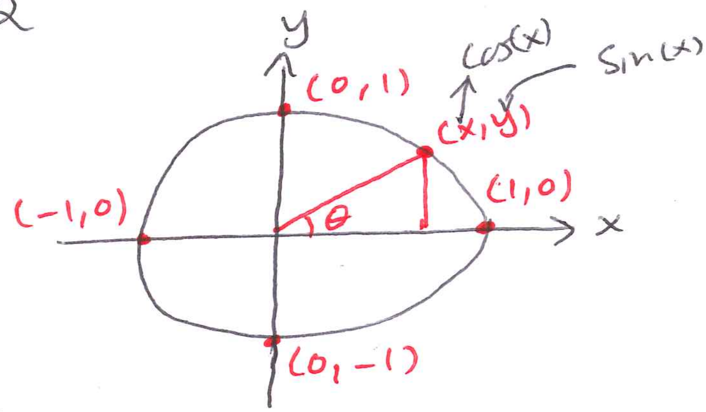
**Part d:** $f(x) = \ln(x-2)$

**Solution:** $x - 2 > 0 \Rightarrow x > 2$ $\boxed{\text{continuous when } x > 2}$.

**Part e:** $f(x) = \tan(x)$

**Solution:** We know that $f(x) = \tan(x) = \frac{\sin(x)}{\cos(x)}$.

It's continuous when $x \neq \frac{\pi}{2} + n\pi$




**Theorem 2: Intermediate Value Theorem (IVT)**

Suppose that $f(x)$ is continuous on $[a,b]$ and $M$ is a number between $f(a)$ and $f(b)$. Then, there exists a number $c$ in $[a,b]$ such that $\boxed{f(c) = M}$