Name: Mohammed Kaabar

Section: - Solution -

Absolute and Conditional Convergence Lab

This section is all about dealing with series of the form $\sum_{k=1}^{\infty}(-1)^{k \text{ or } k+1}a_k$

Thm: Alternating Series Test

An alternating series (written in one of the two forms above) converges if both of the following conditions are met:

1.)

$$a_1 \geq a_2 \geq a_3 \ldots \text{ in other words } \{a_k\} \text{ is a decreasing sequence}$$

2.)

$$\lim_{k\to\infty} a_k = 0$$

Thm: Absolute Convergence Test

If a series converges absolutely then the series converges.

Or

If the series $\sum_{k=1}^{\infty}|u_k|$ converges then the series $\sum_{k=1}^{\infty}u_k$ converges as well.

Remark: This does not work the other way around i.e. just because a series converges does **not** mean the series converges absolutely. Take for instance the convergent series $\sum_{k=1}^{\infty}\frac{(-1)^k}{k}$ and its divergent absolute counterpart, the harmonic series $\sum_{k=1}^{\infty}\frac{1}{k}$. In these cases where a convergent series diverges absolutely we say that the series is **conditionally convergent**. One way to check whether are series diverges when it absolutely diverges is using the following theorem.

Thm: Ratio test for Absolute Convergence

Given a series $\sum_{k=1}^{\infty}u_k$ with

$$\rho = \lim_{k\to\infty}\frac{|u_{k+1}|}{|u_k|}$$

if

a.) $\rho < 1$ then the series converges absolutely and thus converges.

b.) $\rho > 1$ then the series diverges.

c.) $\rho = 1$ then we know nothing more then when we started.

Part 1: Practice with Alternating Series

Directions: Determine if the following series converge absolutely, converge conditionally or diverge.

1. $\sum_{k=1}^{\infty} \frac{(-2)^k k!}{(k+2)!}$ → call it a_n

$$\left|\frac{(-2)^k k!}{(k+2)!}\right| = \frac{2^k k!}{(k+2)(k+1)k!} = \frac{2^k}{(k+2)(k+1)}$$

$$\text{L'H} \lim_{k\to\infty} \frac{k2^{k-1}}{k^2+3k+2} = \frac{k(k-1)2^{k-2}}{2k+3} = \frac{k(k-1)(k-2)2^{k-3}}{2} = \frac{\infty}{2} = \infty \neq 0$$

Since, then the $\sum_{k=1}^{\infty} \frac{(-2)^k k!}{(k+2)!}$ diverges by the divergence test.

2. $\sum_{k=1}^{\infty} (-1)^n 2^{\frac{1}{n}}$

$\lim_{n\to\infty} 2^{\frac{1}{n}} = 2^{\frac{1}{\infty}} = 2^0 = 1 \neq 0$ diverges by the divergence test.

3. $\sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)^3}$

$\frac{1}{(2k+1)^3} \leq \frac{1}{(2k)^3}$ $\quad$ $p = 3 > 1$ converges by Comparison test

So, $\lim_{k\to\infty} \frac{1}{(2k+1)^3} = \frac{1}{\infty} = 0$

it's decreasing.

Hence, it's absolutely convergent.

Part 2: Alternating Series in Action.

So why do we care about absolute versus conditional converge? Why make the distinction to begin with? The answer is rooted in the concept of infinite rearrangement. An infinite rearrangement of an infinite series is a series obtained by changing the order of an infinite amount terms. The rearrangement must eventually use all of the terms of the original infinite series but we are just adding them in a different order! For instance one way to infinitely rearrange a series is to add all of the positive terms first and then all of the negative terms.

The key here is that every rearrangement of an absolutely convergent series has the same sum. This is similar to a familiar property of a finite sum. The sum is the same regardless of the order in which the terms are added. However, this is not true of a conditionally convergent series. Given any real number L, a conditionally convergent series can be rearranged to have sum L! The next part of this lab will explore that amazing fact.

Let's consider a very simple alternating series, $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}$. One can quickly check this series converges conditionally but not absolutely. Now, let's attempt to rearrange in a way the summation approaches 1.5. First, note that every odd term is positive and every even term is negative. Moreover, if we add odd terms only (i.e. $\sum_{k=1}^{\infty}(-1)^{2k}\frac{1}{2k-1}$) it goes to ∞ and if we add even terms only, it goes to $-\infty$. (why?) Therefore, our strategy will be to add odd terms til the sum is bigger than 1.5 and then add even terms til the sum is smaller than 1.5 and repeat this procedure.

1. Let's say we need first n many odd terms so that the sum is bigger than 1.5. (i.e. $\sum_{k=1}^{n}(-1)^{2k}\frac{1}{2k-1} \geq 1.5$) What's n? Compute the sum of first n many odd terms(the sum doesn't need to be exact for our purpose).

   $n=3$

   $1+\frac{1}{3}+\frac{1}{5}=1.583 \approx 1.58$

2. Now, how many even terms do we need to add to the previous answer so that the sum is smaller than 1.5? What's the sum of these even terms added to the previous answer?

   $n=1$

   $1.58-\frac{1}{2}=1.08$

3. Now, how many odd terms do we need to add to the previous answer so that the sum is greater than 1.5? (note that we cannot use first n many odd terms.) What's the sum of these odd terms added to the previous answer? Did our sum get closer to 1.5 compared to step 1?

   $1.08+\frac{1}{7}\approx 1.23$

   $1.23+\frac{1}{9}\approx 1.34$

   $1.34+\frac{1}{11}\approx 1.43$

   $1.43+\frac{1}{13}\approx 1.506 \approx 1.51$ closer to 1.5

4. Again, how many even terms do we need to add to the previous answer so that the sum is smaller than 1.5? (similarly, we can't use even terms used in step 2.) What's the sum of these even terms added to the previous answer? Did our sum get closer to 1.5 compared to step 2? How about compared to step 3?

$1.51 - \frac{1}{4} \approx 1.26$ closer to 1.5 compared to step 2.

5. Let's say you repeated this procedure many times. In each odd step, do you expect your sum to get closer to 1.5? Why? How about even steps? What can you conclude from your prediction?

Any reasonable answer is correct.

Part 3: Extra Credit

Below you will attempt to rearrange the terms of an alternating p-series to approach a given limit L. This will be done with the aid of a web page constructed by Dr. Kevin Cooper for this lab. Log in to my.math.wsu.edu and then go to

http://my.math.wsu.edu/CalcProject/alternating_series.php

Read the test on the web page and set values of p and the target sum to do the following four exercises. Print out the graph you obtain after each step. (you may put all graphs in one page)

1. Use $p = 0.5$ and a target value of 1 to begin to rearrange a conditionally convergent series to approach limit $L = 1$. Start by clicking the + button until the partial sum is greater than L. Then click the − button until the partial sum decreases below L. Then click the + button again, and so forth. Do this for $n = 20$ terms. Does it look like the partial sums will continue to approach L as n increases? Print the graph you obtained after 20 steps.

2. Repeat this exercise using $p = 2$. Print the graph.

3. Use $p = 0.5$ and a target value of 2 to begin to rearrange a conditionally convergent series to have limit $L = 2$. Do this for $n = 20$ terms. Does it look like the partial sums will continue to approach L as n increases? Print the graph.

4. Repeat step 3 with $p = 1$ and using at least $n = 40$ terms. Does it look like the partial sums will continue to approach L as n increases? Print the graph.