Study Guide 3

MATH 172 Lab: Sections 7 and 8

Lab Instructor (TA): Mohammed Kaabar

Student's Name: Mohammed KA Kaabar

Student's ID: – Solution –

Note: *This study guide contains my practice questions that I think will be useful for preparing you for the third exam in Calculus II.*

Question 1: Determine if the series diverges or converges. Be sure to explain which test you use:

$$\sum_{n=1}^{\infty} \frac{2+(-1)^n}{n\sqrt{n}} \longrightarrow n^1 \cdot n^{1/2} = n^{3/2} = n^{1.5}$$

$$1 \leq 2+(-1)^n \leq 3$$

So, compare $\displaystyle\sum_{n=1}^{\infty} \frac{2+(-1)^n}{n\sqrt{n}}$ with $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n^{1.5}}$

So, $\dfrac{2+(-1)^n}{n^{1.5}} \leq \dfrac{3}{n^{1.5}}$ $p=1.5>1$ it's convergent by p-series test.

Question 2: Determine if the series diverges or converges. Be sure to explain which test you use:

$$\sum_{n=1}^{\infty} \underbrace{\left(1+\frac{1}{n}\right)^2 e^{-n}}_{a_n}$$

b_n

$$\sum_{n=1}^{\infty} e^{-n} = \sum_{n=1}^{\infty} \left(\frac{1}{e^n}\right) = \sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n =$$

$$\boxed{\begin{array}{l}\text{Geometric Series} \\ a=\frac{1}{e} \\ r=\frac{1}{e}<1\end{array}}$$

$$= \frac{a}{1-r} = \frac{1/e}{1-1/e} = \frac{1}{e-1} \quad \text{Convergent}$$

OR $\displaystyle\lim_{n\to\infty} \frac{(1+\cancel{1/n}^{\,0})^2}{\cancel{e^n}} \cdot \cancel{e^n} = 1 > 0$ convergent by limit comparison test.

Question 3: Determine whether the following series diverges, converges conditionally, or converges absolutely:

$$\sum_{n=1}^{\infty}(-1)^n \underbrace{\frac{\sqrt{n}+1}{\sqrt{n^2}+1}}_{\to a_n}$$

Compare $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{\sqrt{n+1}}{\sqrt{n^2+1}}$ with $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n}}{\sqrt{n^2}} = \sum_{n=1}^{\infty}\frac{n^{1/2}}{n^1} = \sum_{n=1}^{\infty} n^{1/2-1} =$

$n^{1/2}$, $n^{\frac{2}{2}}$

$\displaystyle = \sum_{n=1}^{\infty} n^{-1/2} = \sum_{n=1}^{\infty}\frac{1}{n^{1/2}}$ $= b_n$ by p-series test $p=1/2<1$ diverges.

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{\sqrt{n+1}}{\sqrt{n^2+1}}\cdot\sqrt{n} = \sqrt{\frac{n(n+1)}{n^2+1}} = \sqrt{\frac{n^2+n}{n^2+1}} = \cancel{\frac{n^2}{n^2}} = 1>0$$

So, it's divergent by limit comparison Test.

① Alternating

② $f(x)=\sqrt{\dfrac{x+1}{x^2+1}} = \left(\dfrac{x+1}{x^2+1}\right)^{1/2} \Rightarrow f'(x) = \dfrac{1}{2}\left(\dfrac{x+1}{x^2+1}\right)^{-1/2}\cdot\left(\dfrac{x^2+1-2x(x+1)}{(x^2+1)^2}\right) = \dfrac{1}{2}\sqrt{\dfrac{x^2+1}{x+1}}\cdot\left(\dfrac{-x^2-2x+1}{(x^2+1)^2}\right) < 0$

③ $\displaystyle\lim_{n\to\infty}\sqrt{\frac{n+1}{n^2+1}} = \sqrt{\frac{\frac{1}{n}+\frac{1}{n^2}}{1+1/n^2}} = \sqrt{\frac{\frac{1}{\infty}+\frac{1}{\infty}}{1+1/\infty}} = \sqrt{\frac{0}{1+0}} = 0$

Convergent by Alternating Series Test. So, it's Conditionally $\boxed{\text{Convergent}}$

Question 4: Determine whether the following series diverges, converges conditionally, or converges absolutely:

$$\sum_{n=1}^{\infty}(-1)^n\left(\sqrt{n+1}-\sqrt{n}\right)$$

$$\sum_{n=1}^{\infty}(-1)^n\left(\sqrt{n+1}-\sqrt{n}\right)\cdot\left(\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}\right)$$

$$=\sum_{n=1}^{\infty}(-1)^n\left(\frac{\cancel{n}+1-\cancel{n}}{\sqrt{n+1}+\sqrt{n}}\right) = \sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{\sqrt{n+1}+\sqrt{n}}\right)$$

Compare $\displaystyle\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{\sqrt{n+1}+\sqrt{n}}\right)$ with $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ $p=1/2<1$ divergent. By limit Comparison Test,

$$\lim_{n\to\infty}\frac{\frac{1}{\sqrt{n+1}+\sqrt{n}}}{1/\sqrt{n}} = \lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{2}>0$$

So, it's divergent.

By Alternating Series test, it's decreasing, convergent. Hence, It's conditionally convergent.

Question 5: Use a comparison test to determine whether the integral converges or diverges:

$$\int_1^{\infty} \frac{2 + \sec^2 x}{x}\,dx \qquad \boxed{\int_1^{\infty} \frac{1}{x^p}\,dx}$$

$$\frac{2+\sec^2(x)}{x} \geq \frac{2}{x}$$

$$\int_1^{\infty} \frac{2+\sec^2(x)}{x}\,dx \geq \int_1^{\infty} \frac{2}{x}\,dx$$

diverges $p=1$ p-series test

diverges $p=1$ p-series test

Since both of them diverge, then by direct comparison test $\int_1^{\infty} \frac{2+\sec^2 x}{x}\,dx$ diverges.

Question 6: Find the sum of the following series:

$$\sum_{k=1}^{\infty} \frac{4}{k(k+2)}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{4}{k(k+2)}$$

$\Rightarrow$ By partial fractions: Cover Method

$$\boxed{\frac{4}{k(k+2)} = \underbrace{\frac{2}{k}}_{k=0} + \underbrace{\frac{-2}{k+2}}_{k=-2}}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{\infty} \left(\frac{2}{k} - \frac{2}{k+2}\right)$$

$$\Rightarrow \sum_{k=1}^{\infty} \left(\frac{2}{k} - \frac{2}{k+2}\right) = \left(2 - \cancel{\frac{2}{3}}\right) + \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{2}{3}} - \cancel{\frac{2}{5}}\right) + \left(\cancel{\frac{1}{2}} - \frac{1}{3}\right) + \cdots +$$

$$\left(\cancel{\frac{2}{n-2}} - \cancel{\frac{2}{n}}\right) + \left(\cancel{\frac{2}{n-1}} - \frac{2}{n+1}\right) + \left(\cancel{\frac{2}{n}} - \cancel{\frac{2}{n+2}}\right) = \left(2 + 1 - \frac{2}{n+1}\right)$$

$$\lim_{n\to\infty} \left(2 + 1 - \cancel{\frac{2}{n+1}}^{\,0}\right) = 3.$$

Question 7: Determine whether the following sequence is increasing or decreasing:

$$\boxed{a_n = \frac{3^n}{(n+2)!}}$$

$$a_{n+1} = \frac{3^{n+1}}{(n+3)!}$$

$$\frac{a_{n+1}}{a_n} = \frac{\cancel{3^{n+1}}\ 3^1}{\cancel{(n+3)!}\ (n+3)} \cdot \frac{\cancel{(n+2)!}}{\cancel{3^n}} = \frac{3}{n+3} \leq 1 \text{ for } n \geq 0$$

So, it's decreasing because $a_{n+1} \leq a_n$.

Question 8: Determine whether the following sequence converges or diverges:

$$a_n = \frac{e^n + 2}{e^{2n} - 1} \quad \text{leading terms}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{e^n}{e^{2n}} \overset{L'H}{=} \lim_{n\to\infty} \frac{e^n}{2e^{2n}} = \lim_{n\to\infty} \frac{1}{2e^n} = \frac{1}{2e^\infty} = \frac{1}{\infty}$$

$= 0$ Convergent.

Question 9: Show the following sequence is bounded:

$$a_n = \frac{3n^2 - 2}{n^2 + 1}$$

$$\frac{3n^2 - 2}{n^2+1} \leq \frac{3n^2}{n^2+1} \leq \frac{3\cancel{n^2}}{\cancel{n^2}} = 3$$

$$\boxed{\begin{gathered} n^2+1 > n^2 \\ \frac{1}{n^2+1} < \frac{1}{n^2} \end{gathered}}$$

$$\Downarrow$$

$$|a_n| \leq 3$$

So, it's bounded by 3.

Question 10: Use the integral test to determine the convergence or divergence of the following series:

$$\sum_{m=1}^{\infty} \frac{e^{\frac{1}{m}}}{m^2}$$

$f(x) = \dfrac{e^{\frac{1}{x}}}{x^2}$, f(x) positive, continuous everywhere on $[1, \infty)$

To show decreasing: $f'(x) = \dfrac{\left(\cancel{x^2} \cdot \frac{-1}{\cancel{x^2}} e^{\frac{1}{x}}\right) - \left(2x \cdot e^{1/x}\right)}{x^4} = \dfrac{-e^{\frac{1}{x}} - 2x e^{\frac{1}{x}}}{x^4} =$

$= \dfrac{-e^{\frac{1}{x}}(1+2x)}{x^4} < 0 \Rightarrow$ So, f(x) is decreasing.

$\displaystyle\int_1^{\infty} \frac{e^{1/x}}{x^2}\,dx = \lim_{t\to\infty} \boxed{\int_1^t \frac{e^{1/x}}{x^2}\,dx}$

let $u = \dfrac{1}{x}$

$du = \dfrac{-1}{x^2}dx \Rightarrow \dfrac{dx}{x^2} = -du$

$\displaystyle\int_1^t \frac{-e^u}{1}\,du = -\int_1^t e^u\,du = -e^u\Big|_1^t = -e^{1/x}\Big|_1^t =$

$= -e^{1/t} + e^1$

So, $\displaystyle\lim_{t\to\infty}\left(-e^{1/t} + e^1\right) = -e^{1/\infty} + e^1 = -\cancel{e}^{\,1} + e^1 = \boxed{e-1}$ convergent by integral test.

Question 11: Determine whether the series converges or diverges. If convergent, find the sum of the series:

$$\sum_{m=2}^{\infty}\left(\frac{2}{3^m} + \frac{1}{2^m}\right) = \sum_{m=2}^{\infty}\left(\frac{2}{3^m}\right) + \sum_{m=2}^{\infty}\left(\frac{1}{2^m}\right) =$$

$$= \sum_{m=2}^{\infty} 2\left(\frac{1}{3^m}\right) + \sum_{m=2}^{\infty} 1\left(\frac{1}{2^m}\right) = \sum_{m=2}^{\infty} 2\left(\frac{1}{3}\right)^m + \sum_{m=2}^{\infty} 1\left(\frac{1}{2}\right)^m$$

$a = 2\left(\frac{1}{3}\right)^2 = 2\left(\frac{1}{9}\right) = \frac{2}{9}$

$r = \frac{1}{3} < 1$

Convergent

$S_1 = \dfrac{a}{1-r} = \dfrac{2/9}{1-1/3} = \dfrac{2/9}{2/3} = \boxed{\dfrac{1}{3}}$

$a = 1\left(\frac{1}{2}\right)^2 = 1\left(\frac{1}{4}\right) = \frac{1}{4}$

$r = \frac{1}{2} < 1$

Convergent

$S_2 = \dfrac{a}{1-r} = \dfrac{1/4}{1-1/2} = \dfrac{1/4}{1/2} = \boxed{\dfrac{1}{2}}$

Thus, $S_n = S_1 + S_2 = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$ ∴ Convergent

Question 12: Prove that $0.1\bar{9} = 0.2$.

$$0.1\bar{9} = 0.19999$$

$$= 0.1 + 0.09 + 0.009 + 0.0009 + \cdots$$

$$= 0.1 + \frac{9}{100} + \frac{9}{1000} + \frac{9}{10{,}000} + \cdots$$

$$a = \frac{9}{100}$$

$$r = \frac{1}{10} < 1$$

$$= 0.1 \left[\frac{a}{1-r} = \frac{9/100}{1 - 1/10} = \frac{1}{10} \right]$$

$$= 0.1 + 0.1 = \boxed{0.2}$$

$$\boxed{\text{So, } 0.1\bar{9} = 0.2}$$

Question 13: Show that the series $\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{5^n}$ is convergent. How many terms are needed to approximate the series with the maximum error of $3(10)^{-2}$.

Part ⓐ

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{5^n}$$

① Alternating

② $\lim_{n\to\infty} \frac{n^2}{5^n} \overset{L'H}{=} \lim_{n\to\infty} \frac{2n}{5^n \ln(5)} \overset{L'H}{=} \lim_{n\to\infty} \frac{2}{5^n (\ln 5)^2} \quad \frac{2}{\infty} = 0$

③ $\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{\cancel{5^{n+1}}_{5^1}} \cdot \frac{\cancel{5^n}}{n^2} = \frac{(n+1)^2}{5n^2} = \frac{(n+1)^2}{5n^2} < 1$ decreasing.

So, it's convergent by Alternating Series Test.

Part ⓑ $|\text{Error}| = |S - S_n|$

$$|\text{Error}| \leq a_{n+1} = \frac{(n+1)^2}{5^{n+1}} \leq \frac{3}{10^2}$$

$$\frac{(n+1)^2}{5^{n+1}} \leq \frac{3}{10^2}$$

$$\frac{5^{n+1}}{(n+1)^2} \geq \frac{10^2}{3} \Rightarrow \frac{5^{n+1}}{(n+1)^2} \geq 33.33$$

let's assume $\boxed{n = 3} \Rightarrow \frac{5^{3+1}}{(3+1)^2} = 34.06 \geq 33.33$

Therefore, when Approximately $\boxed{n = 3}$ or more, the series will be greater than $\frac{10^2}{3} \approx 33.33$

Question 14: Determine the radius and interval of convergence for the series:

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x-1)^n}{(2n+3)4^n} \Rightarrow \sum_{n=1}^{\infty} \frac{(x-1)^n}{(2n+3)4^n}$$

By Ratio Test

$$= \lim_{n\to\infty} \left| \frac{\cancel{(x-1)^{n+1}}^{(x-1)}}{(2n+5)\cancel{4^{n+1}}_{4^1}} \cdot \frac{(2n+3)\cancel{4^n}}{\cancel{(x-1)^n}} \right| = \lim_{n\to\infty} \left| \frac{(x-1)(2n+3)}{4(2n+5)} \right| =$$

$$= \left|\frac{x-1}{4}\right| \lim_{n\to\infty} \frac{2n+3}{2n+5} = \left|\frac{x-1}{4}\right| < 1 \text{ Converges}$$

diverges converges

$$= |x-1| < 4 \Rightarrow -4 < x-1 < 4 \Rightarrow \boxed{-3 < x \leq 5}$$

$\boxed{\text{Radius} = 4}$ and Interval of Convergence (IC): $(-3, 5]$

$x = -3$	$x = 5$
$\sum_{n=1}^{\infty} \frac{(-1)^n(-4)^n}{(2n+3)4^n}$	$\sum_{n=1}^{\infty} \frac{(-1)^n \cancel{(4)^n}}{(2n+3)\cancel{4^n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+3)}$
$= \sum_{n=1}^{\infty} \frac{(-1)^n (-1)^n \cancel{4^n}}{(2n+3)\cancel{4^n}}$	(i) Alternating Series
$= \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{(2n+3)}$	(ii) $\lim_{n\to\infty} \frac{1}{2n+3} = \frac{1}{\infty} = 0$
$= \sum_{n=1}^{\infty} \frac{1}{2n+3}$	(iii) $f(x) = \frac{1}{2x+3} = (2x+3)^{-1}$
Compare it with	$f'(x) = -(2x+3)^{-2} \cdot (2)$
$\sum_{n=1}^{\infty} \frac{1}{2n} \rightarrow$ p-series $p=1$ Diverges	$= \frac{-2}{(2x+3)^2} < 0$ decreasing
$\lim_{n\to\infty} \frac{2n}{2n+3} = 1 > 0$	So, it's convergent by Alternating Series Test.
diverges by limit Comparison test.	

Question 15:

Part a: Find a power series representation about $x = 0$ for $f(x) = \frac{1}{5+4x}$. Then find the radius of convergence.

$$\frac{1}{5+4x} = \frac{1}{5} \cdot \frac{1}{1+\left(\frac{4}{5}x\right)}$$

$$= \frac{1}{5} \cdot \frac{1}{1-\left(\frac{-4}{5}x\right)}$$

$$\boxed{\begin{array}{l} a = 1 \\ r = \frac{-4}{5}x \end{array}} \quad = \frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{-4}{5}x\right)^n$$

$$\boxed{= \frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{-4}{5}\right)^n x^n}$$

Root Test: $\sum |\text{term}|$

$$\left|\frac{-4}{5}x\right| = \left|\frac{4}{5}x\right| < 1 \quad \text{Converges}$$

$$|x| < \frac{5}{4}$$

$$-\frac{5}{4} < x < \frac{5}{4}$$

$$\boxed{\text{Radius} = \frac{5}{4}}$$

Part b: Use part a, to find a power series representation for $g(x) = -\frac{4}{(5+4x)^2}$.

By differentiating both sides:

$$\frac{1}{5+4x} = [5+4x]^{-1} = -[5+4x]^{-2} \cdot (4) = \frac{-4}{(5+4x)^2}$$

$$\frac{-4}{(5+4x)^2} = \frac{1}{5} - \frac{4}{25}x + \frac{16}{125}x^2 + \cdots \leftarrow \text{without differentiating}$$

$$\frac{-4}{(5+4x)^2} = 0 - \frac{4}{25} + \frac{32}{125}x + \cdots \leftarrow \text{After differentiating}$$

$$\frac{-4}{(5+4x)^2} = 0 - \frac{4}{25} + \frac{32}{125}x + \cdots = \boxed{\frac{1}{5}\sum_{n=1}^{\infty}\left(\frac{-4}{5}\right)^n n x^{n-1}}$$

Part c: Use part b, to find a power series representation for $h(x) = \ln(5 + 4x)$.

$$\int_0^x \left(\frac{1}{5+4x}\right)dx = \int_0^x \left[\frac{1}{5} - \frac{4}{25}x + \frac{16}{125}x^2 + \cdots\right]dx \quad \text{by integrating both sides}$$

$$\frac{1}{4}\ln(5+4x)\Big|_0^x = \left[\frac{1}{4}\ln(5+4x) - \frac{1}{4}\ln(5)\right] \leftarrow \text{Left-hand Side}$$

For Right-hand Side, $= \frac{1}{5}x - \frac{2}{25}x^2 + \frac{16}{375}x^3 + \cdots = \sum_{n=0}^{\infty}\left(\frac{-4}{5}\right)^n \frac{x^{n+1}}{n+1}$

So, $\ln(5+4x) = 4\left[\left(\frac{1}{4}\ln(5)\right) + \left(\frac{1}{5}x - \frac{2}{25}x^2 + \frac{16}{375}x^3 + \cdots\right)\right]$

$$= \left[\ln(5) + \frac{4}{5}x - \frac{8}{25}x^2 + \frac{64}{375}x^3 + \cdots\right] = \ln(5) + 4\sum_{n=0}^{\infty}\left(\frac{-4}{5}\right)^n \frac{x^{n+1}}{n+1}$$

Question 16: How many terms of the Maclaurin series for $\ln(x+1)$ do you need to use to estimate $\ln(1.4)$ within 0.001?

$$|\text{Error}| \le \left| \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \right| \le 0.001$$

Since $x+1=1.4$, then $x = 1.4-1 = \boxed{0.4}$

$n=?$

$f(x) = \ln(x+1) \longrightarrow f(0) = \ln(0+1) = 0$

$f'(x) = \dfrac{1}{x+1} = (x+1)^{-1} \longrightarrow f'(0) = 1$

$f''(x) = -(x+1)^{-2} \longrightarrow f''(0) = -1$

$f'''(x) = 2(x+1)^{-3} \longrightarrow f'''(0) = 2$

$f^{(4)}(x) = -6(x+1)^{-4} \longrightarrow f^{(4)}(0) = -6$

$$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$$

$$\left| f^{(n+1)}(c) \right| = \left| \frac{(n)!}{(1+c)^{n+1}} \right|$$

$$|\text{Error}| \le \left| \frac{n!\,(0.4)^{n+1}}{(1+c)^{n+1}(n+1)!} \right| \le 0.001$$

$$\left| \frac{(0.4)^{n+1}}{(1+c)^{n+1}(n+1)} \right| \le 0.001$$

$$\left| \frac{(0.4)^{n+1}}{n+1} \right| \le 0.001$$

$\boxed{n=5}$.

Good Luck in Exam 3

Best of Luck

Mohammed K A Kaabar