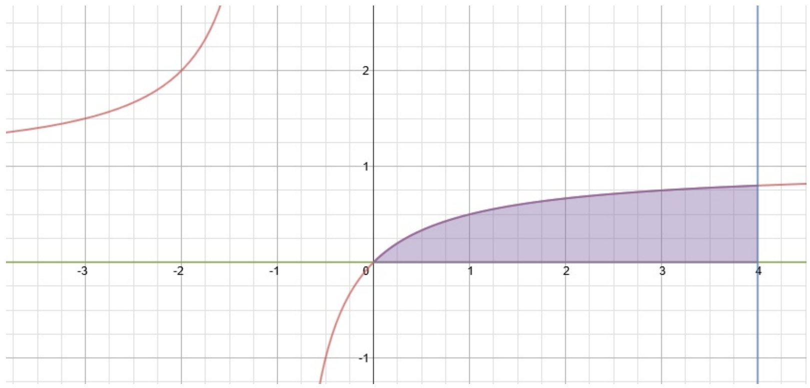
Written Homework 1 Solutions

§ **7.5 #57**: First, we draw a picture of the bounded region:



Thus, the volume is given by

$$V = \pi \int_0^4 \frac{x^2}{(x+1)^2}\, dx.$$

There are a few different ways to solve this integral. We will show two methods – the first using long division and partial fractions, and the second using u-substitution.

<u>Method 1:</u>

$$\pi \int_0^4 \frac{x^2}{(x+1)^2}\, dx = \pi \int_0^4 \frac{(x+1)^2 - 2x - 1}{(x+1)^2}\, dx$$
$$= \pi \int_0^4 \left(1 - \frac{2x+1}{(x+1)^2}\right) dx.$$

Note we could have gotten this result by doing long division as well (this is basically the same thing). Now, we have to use partial fractions. We have

$$\frac{2x+1}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} = \frac{A(x+1)+B}{(x+1)^2}.$$

So

$$2x + 1 = A(x+1) + B = Ax + A + B.$$

Thus, equating coefficients, we get $A = 2$ and $A + B = 1$. So $B = 1 - A = -1$. Hence,

$$\frac{2x+1}{(x+1)^2} = \frac{2}{x+1} + \frac{-1}{(x+1)^2}$$

Going back to our integral, we have

$$\begin{aligned}
\pi \int_0^4 \frac{x^2}{(x+1)^2}\, dx &= \pi \int_0^4 \left(1 - \left(\frac{2}{x+1} + \frac{-1}{(x+1)^2}\right)\right)\, dx \\
&= \pi \int_0^4 \left(1 - \frac{2}{x+1} + \frac{1}{(x+1)^2}\right)\, dx \\
&= \pi \left(x - 2\ln|x+1| - \frac{1}{x+1}\right)\Big|_0^4 \\
&= \pi \left(4 - 2\ln(5) - \frac{1}{5} - 0 + 2\ln(1) + 1\right) \\
&= \pi \left(\frac{24}{5} - \ln(25)\right)
\end{aligned}$$

<u>Method 2:</u> We compute the integral using u-substitution. Let

$$u = x+1,\ du = dx.$$

Note $x = u - 1$. Thus,

$$\begin{aligned}
\pi \int_0^4 \frac{x^2}{(x+1)^2}\, dx &= \pi \int_1^5 \frac{(u-1)^2}{u^2}\, du \\
&= \pi \int_1^5 \frac{u^2 - 2u + 1}{u^2} \\
&= \pi \int_1^5 \left(1 - \frac{2}{u} + \frac{1}{u^2}\right)\, du \\
&= \pi \left(u - 2\ln|u| - \frac{1}{u}\right)\Big|_1^5 \\
&= \pi \left(5 - 2\ln(5) - \frac{1}{5} - 1 + 2\ln(1) + 1\right) \\
&= \pi \left(\frac{24}{5} - \ln(25)\right).
\end{aligned}$$

§ **7.8 #62**: The Fundamental Theorem by Calculus (FTC) can only be applied when the function we are integrating is continuous over the interval of integration. However, $f(x) = \frac{1}{x}$ is not continuous on the interval $[-1, 1]$. Note

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

and

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty.$$

Thus, the FTC cannot be applied.

§ **7.8 #66**: Note

$$\begin{aligned}
\int_0^\infty xe^{-x}\,dx &= \begin{vmatrix} u = x & dv = e^{-x}\,dx \\ du = dx & v = -e^{-x} \end{vmatrix} \\
&= -xe^{-x}\Big|_0^\infty + \int_0^\infty e^{-x}\,dx \\
&= \left(-xe^{-x} - e^{-x}\right)\Big|_0^\infty \\
&= \lim_{t\to\infty}\left(-te^{-t} - e^{-t} + 0 + e^0\right) \\
&= 1.
\end{aligned}$$