**Written Homework 1 Solutions**

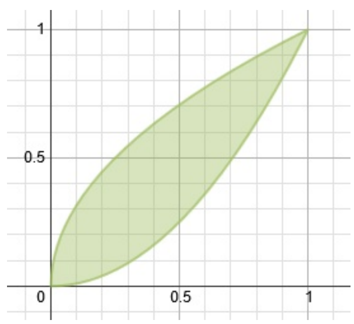
§ **6.2 #59**: Note

$$\begin{aligned} A_n &= \int_0^1 \left(x^{1/n} - x^n\right)\, \mathrm{d}x \\ &= \left(\frac{1}{\frac{1}{n}+1} x^{1/n+1} - \frac{1}{n+1} x^{n+1}\right)\Big|_0^1 \\ &= \frac{n}{n+1} - \frac{1}{n+1} \\ &= \frac{n-1}{n+1}. \end{aligned}$$

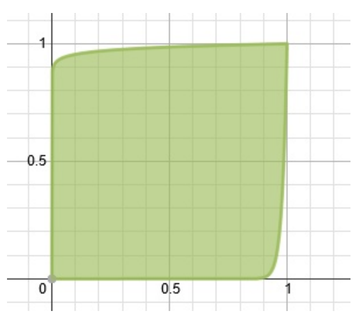
So

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} \frac{n-1}{n+1} = 1.$$

What is happening is, as $n$ gets really large, the area bounded by the curves is approaching the area of a square of length 1. To see this, look at first the graph of the area between $y = x^{1/2}$ and $y = x^2$:




Now, let us look at the graph of the area between $y = x^{1/50}$ and $y = x^{50}$.

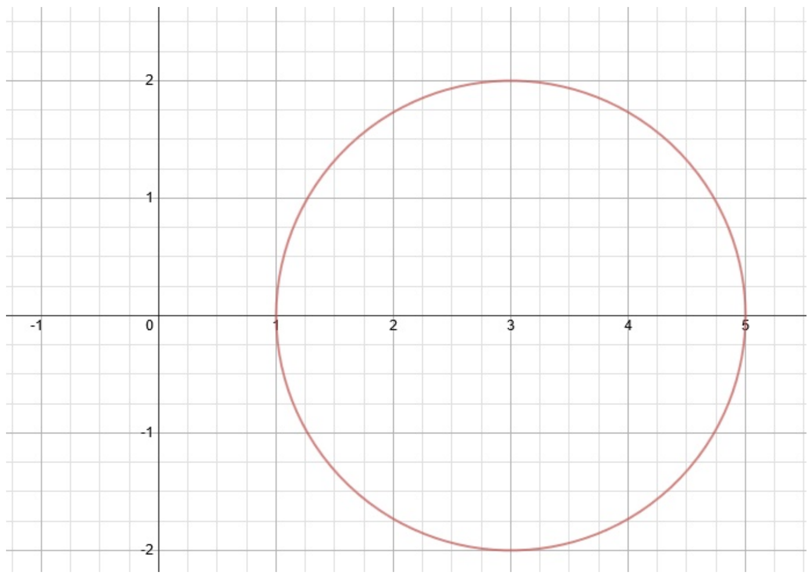



§ **6.3 #16**: Take a cross-section of our volume. Each cross-section will be a circle with area $\pi r^2$. Hence, we just need to sum up all the circles by integrating. We get

$$V = \int_0^h \pi r^2 \mathrm{d}y \;=\; \pi r^2 h.$$

Note the 45° angle doesn't affect the volume!

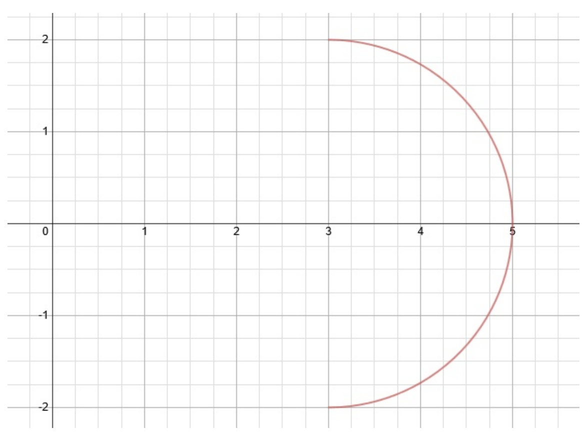
§ **6.3 #67**: Let us first draw a picture of our circle $(x-3)^2 + y^2 + 4$. This is a circle centered at $(3, 0)$ with radius 2.



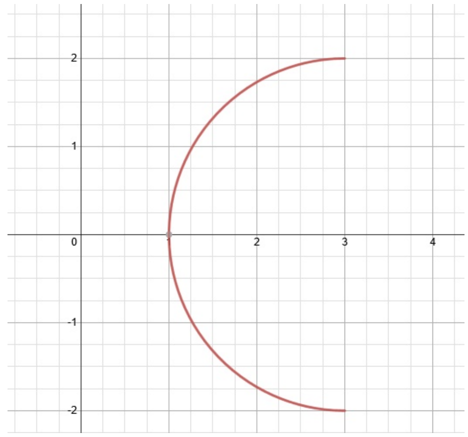

Solving the equation $(x-3)^2 + y^2 = 4$ for $x$, we get

$$x = 3 \pm \sqrt{4 - y^2}.$$

The equation with the + sign, $x = 3 + \sqrt{4 - y^2}$, is the equation for the right part of the circle, as shown below:




Similarly, $x = 3 - \sqrt{4 - y^2}$ describes the left side of the circle, as shown below.




Now, to find our volume, we use the washer method. Each washer will have an inner radius and an outer radius given by

$$R_{\text{in}} = 3 - \sqrt{4 - y^2}, \qquad R_{\text{out}} = 3 + \sqrt{4 - y^2}.$$

So

$$\begin{aligned} V &= \pi \int_{-2}^{2} \left[ \left(3 + \sqrt{4 - y^2}\right)^2 - \left(3 - \sqrt{4 - y^2}\right)^2 \right] \mathrm{d}y \\ &= \pi \int_{-2}^{2} \left[ 9 + 6\sqrt{4 - y^2} + 4 - y^2 - \left(9 - 6\sqrt{4 - y^2} + 4 - y^2\right) \right] \mathrm{d}y \\ &= 12\pi \int_{-2}^{2} \sqrt{4 - y^2}\, \mathrm{d}y. \end{aligned}$$

Now, since we don't know how to solve this integral yet, we use geometry instead. Note

$$x^2 + y^2 = 2^2$$

is the equation of a circle of radius 2. Solving for $x$ gives

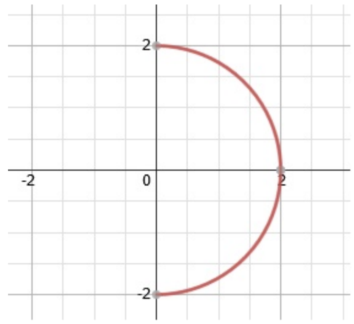
$$x = \pm\sqrt{4 - y^2}$$

So

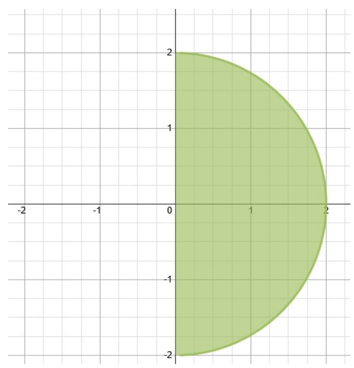
$$x = +\sqrt{4 - y^2}$$

is the equation for a semicircle where $x > 0$.

See picture below.




Note $\sqrt{4-y^2}$ is exactly what we have in our integral. We are integrating from $y=-2$ to $y=2$ so the integral represents the area shown below.



This area is just $\frac{1}{2}\pi(2)^2$ (one half the area of a circle with radius 2). So

$$\begin{aligned} V &= 12\pi \int_{-2}^{2} \sqrt{4-y^2}\, \mathrm{d}y \\ &= 12\pi \left( \frac{1}{2}\pi(2)^2 \right) \\ &= 24\pi^2. \end{aligned}$$

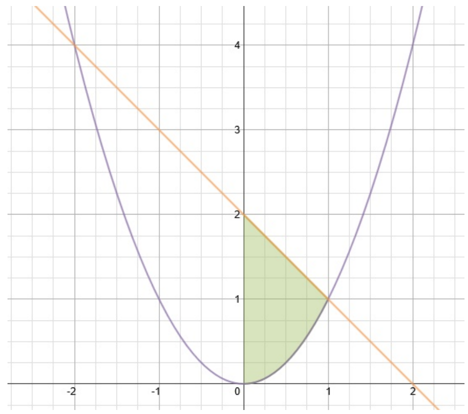
§ **6.3 #42**: First, let's find where the curves intersect. Set

$$x^2 = 2 - x.$$

Thus,

$$x^2 + x - 2 = 0 \quad \Rightarrow \quad (x+2)(x-1) = 0 \quad \Rightarrow \quad x = -2,\ x = 1.$$

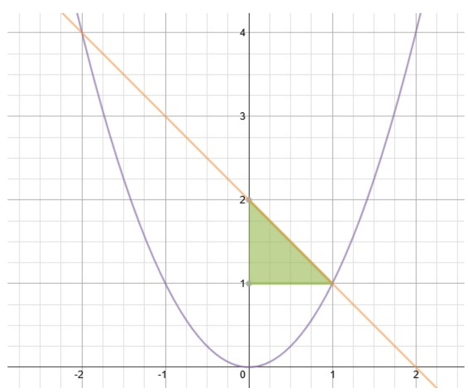
Now let's draw a picture and shade the area we want to rotate.




We can use the shell method with cylindrical shells of radius $x$ and height $2 - x - x^2$. Thus,

$$
\begin{aligned}
V &= 2\pi \int_0^1 x(2 - x - x^2)\, \mathrm{d}x \\
&= 2\pi \int_0^1 (2x - x^2 - x^3)\, \mathrm{d}x \\
&= 2\pi \left( x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 \right) \Big|_0^1 \\
&= 2\pi \left( 1 - \frac{1}{3} - \frac{1}{4} \right) \\
&= \frac{5\pi}{6}.
\end{aligned}
$$

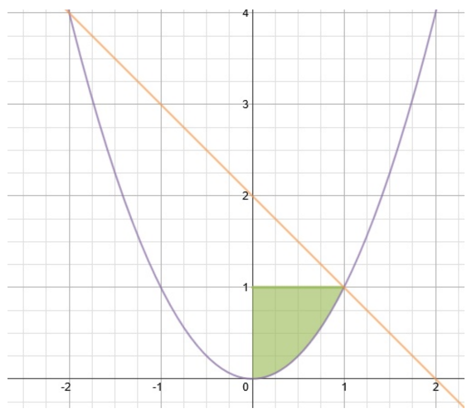
We can also use the disk method, but we have to use two integrals. First, we have to rotate the region above $y = 1$, i.e. the region shaded below.




The volume we get when we rotate this region about the $y$-axis, using the disk method (each disk has a radius of $2-y$), is given by

$$\begin{aligned}
V_1 &= \pi \int_1^2 (2-y)^2 \, \mathrm{d}y \\
&= \pi \int_1^2 (4-4y+y^2) \, \mathrm{d}y \\
&= \pi (4y - 2y^2 + \frac{1}{3}y^3)\Big|_1^2 \\
&= \pi \left(8 - 8 + \frac{8}{3} - 4 + 2 - \frac{1}{3}\right) \\
&= \pi \left(\frac{7}{3} - 2\right) \\
&= \frac{\pi}{3}.
\end{aligned}$$

Now we rotate the region below $y=1$, i.e. the region shaded below.




Using the disk method (each disk has a radius $\sqrt{y}$), the volume we get when we rotate this region about the $y$-axis is given by

$$\begin{aligned}
V_2 &= \pi \int_0^1 \left(\sqrt{y}\right)^2 \, \mathrm{d}y \\
&= \pi \int_0^1 y \, \mathrm{d}y \\
&= \frac{1}{2}\pi.
\end{aligned}$$

Therefore, the total volume we get when we rotate our desired region about the $y$-axis is given by

$$V = V_1 + V_2 = \frac{\pi}{3} + \frac{\pi}{2} = \frac{5\pi}{6}.$$

The shell method was definitely easier!