Math 172: Second Semester Calculus

Fall Semester, 2014

Final Exam

#	1	2	3	4	5	6	7	subtotal
worth	11	11	11	11	11	10	11	

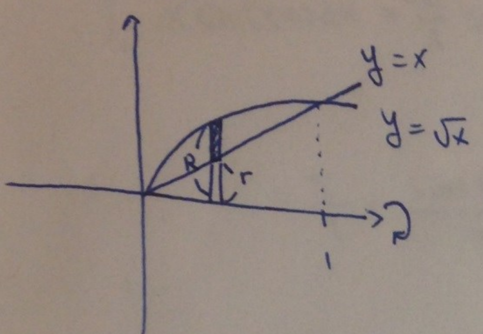
#	8	9	10	11	12	13	14	subtotal	total
worth	10	10	11	11	11	10	11		

Name: ______________________________ Section: ________

Work must be shown to receive credit. No calculators.

Solutions

Question 1. ***(11 points).*** Find the volume of the following solid of revolution: the region bounded by $y_1 = \sqrt{x}$ and by $y_2 = x$ revolved about the x-axis.



$R = x$

$r = \sqrt{x}$

$$V = \pi \int_0^1 (R^2 - r^2)\, dx$$

$$= \pi \int_0^1 (x - x^2)\, dx$$

$$= \pi \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{\pi}{6}$$

Question 2. Evaluate the following indefinite and definite integrals.

a. (**6 points**) $\int_0^{\pi/3} x \sin(3x)\, dx$

$$\int_0^{\pi/3} x \sin(3x)\,dx = \frac{-x}{3}\cos(3x)\Big|_0^{\pi/3} + \int_0^{\pi/3} \frac{\cos(3x)}{3}\,dx$$

$$= \frac{\pi}{9} + \frac{\sin(3x)}{9}\Big|_0^{\pi/3} = \frac{\pi}{9}$$

b. (**5 points**) $\int \frac{6}{x-x^2}\, dx$

$$= 6\int \frac{-1}{x(x-1)}\,dx = 6\int \left[\frac{A}{x} + \frac{B}{1-x}\right] dx$$

$$A - Ax + Bx = 1 \Rightarrow A=1, B=1$$

$$6\int \left[\frac{1}{x} + \frac{1}{1-x}\right] dx = 6\left[\ln|x| - \ln|1-x|\right] + C$$

Question 3a. (**5 points**) Evaluate the infinite series:

$$\sum_{k=0}^{\infty} 2\,e^{-k}$$

$\sum 2e^{-k} = 2\sum e^{-k}$ $\quad |e^{-1}|<1 \Rightarrow$ use geometric series:

$\sum_{k=0}^{\infty} e^{-k} = \frac{1}{1-1/e} = \frac{e}{e-1}$. Thus, $\sum 2e^{-k} = \frac{2e}{e-1}$.

b. (**6 points**) Use a convergence test of your choice to determine whether the following series converges or diverges.

$$\sum_{k=1}^{\infty} \frac{100}{\sqrt{k^3+1}}$$

Div test fails.

Using comparison test: $\frac{100}{\sqrt{k^3+1}} \sim \frac{1}{\sqrt{k^3+1}} < \left(\frac{1}{k+1}\right)^{3/2}$

By P-series test, this converges. So

$\sum \frac{100}{\sqrt{k^3+1}}$ conv. as well

Question 4. (***11 points***) Determine whether the following series is absolutely convergent, conditionally convergent or divergent.

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{2/3}}$$

By alternating series test, this conv. since $\frac{1}{k^{2/3}} \longrightarrow 0$.

$\sum \left| \frac{(-1)^k}{k^{2/3}} \right| = \sum \left(\frac{1}{k}\right)^{2/3}$ div. using p-series test.

(or Integral)

So the series conv. cond.

Question 5. (**11 points**) Let $f(x) = \frac{x}{1+x^2}$ and let $\frac{x}{1+x^2} = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots$ be its Maclaurin series expansion (also known as its Taylor series expansion about 0). Find the coefficients c_0, c_1, c_2 and c_3.

$f(x) = \frac{x}{1+x^2} = x(1+x^2)^{-1}$

$f'(x) = (1+x^2)^{-1} - x(1+x^2)^{-2} \cdot 2x = \underline{(1+x^2)^{-1}} - 2x^2(1+x^2)^{-2}$

$f''(x) = -(1+x^2)^{-2} \cdot 2x - 4x(1+x^2)^{-2} + 4x^2(1+x^2)^{-3} \cdot 2x = \underline{-6x(1+x^2)^{-2}} + \underline{8x^3(1+x^2)^{-3}}$

$f'''(x) = -6(1+x^2)^{-2} + 12x(1+x^2)^{-3} \cdot 2x + 8 \cdot 3x^2(1+x^2)^{-3} - 24x^3(1+x^2)^{-4} \cdot 2x$.

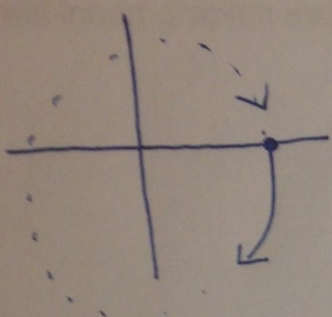
$\Rightarrow$ $f(0) = 1$ $\qquad c_0 = \frac{f(0)}{0!} = 1$

$f'(0) = 1$ $\qquad c_1 = \frac{f'(0)}{1!} = 1$

$f''(0) = 0$ $\qquad c_2 = \frac{f''(0)}{2!} = 0$

$f'''(0) = -6$. $\qquad c_3 = \frac{f'''(0)}{3!} = -1$.

Question 6. (***10 points***) Give parametric equations that describe a full circle of radius 2, centered on the origin with clockwise orientation, where the parameter t varies over the interval $[0,2]$. Assume that the circle starts at the point $(x, y) = (2,0)$.



Since $x = f(t)$, $f(0) = 2$,

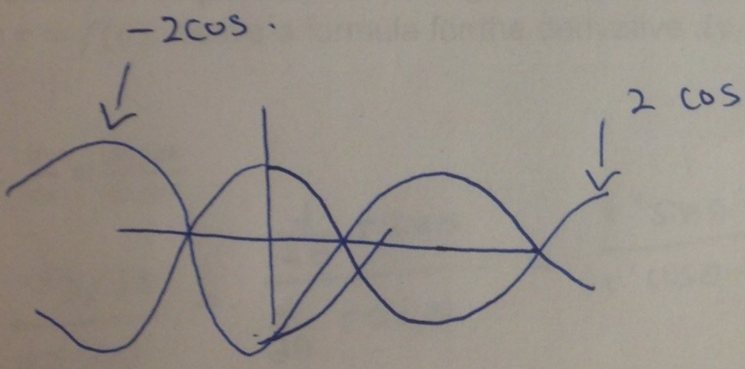
$f(t) = 2\cos(\pm \pi t)$ ← Even and

$g(t) = 2\sin(\pm \pi t)$.

clockwise means $f'(t), g'(t) \leq 0$ at $0 \leq t \leq$

$f'(t) = \mp 2\pi \sin(\pm \pi t)$ ← Either works.

$g'(t) = \pm 2\cos(\pm \pi t)$.



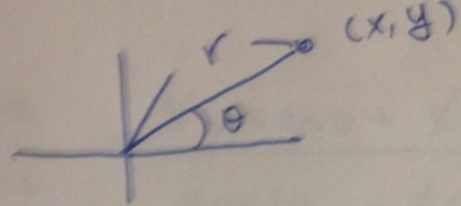
So $g(t) = 2\sin(-\pi t)$

$(x, y) = (2\cos(\pi t), -2\sin(\pi t))$.

Question 7a. (5 points) Write equations for how x and y depend on r and θ in polar coordinates on the graph below.

$$x = r\cos\theta \qquad y = r\sin\theta .$$

[will insert graph here]



Question 7b. (6 points) Assuming that a polar equation is given in the form $r = f(\theta)$, derive a formula for the derivative dy/dx.

Hint: $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$

$$\frac{dy/d\theta}{dx/d\theta} = \frac{\frac{d}{d\theta} r\sin\theta}{\frac{d}{d\theta} r\cos\theta} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta} .$$

Question 8. (***10 points***) Find the slope of the graph of the curve $r = 5\sin(\theta)$ in the xy-plane at the point $(r,\theta) = (\frac{5}{2}, \frac{\pi}{6})$.

Hints: For the graph of $r = f(\theta)$, $\frac{dy}{dx} = \frac{f'(\theta)\sin(\theta)+f(\theta)\cos(\theta)}{f'(\theta)\cos(\theta)-f(\theta)\sin(\theta)}$,

and $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$, $\cos\left(\frac{\pi}{6}\right) = \sqrt{3}/2$.

$r' = 5\cos\theta$.

$$\frac{dy}{dx} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta} = \frac{\cancel{5}\sin\theta\cos\theta + \cancel{5}\sin\theta\cos\theta}{\cancel{5}\cos^2\theta - \cancel{5}\sin^2\theta}$$

$$= \frac{2\cos\theta\sin\theta}{1-2\sin^2\theta} = \frac{2\cdot\frac{1}{2}\cdot\frac{\sqrt{3}}{2}}{1-2\cdot\frac{1}{4}}$$

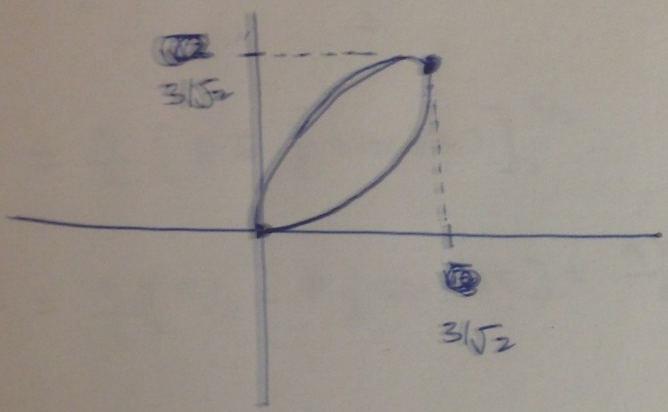
$$= \frac{\sqrt{3}/2}{1/2} = \sqrt{3}.$$

Question 9. (**10 points**) Draw the graph of the polar curve $r = 3\sin(2\theta)$, $0 \leq \theta \leq \pi/2$, in the xy-plane. Use tick marks at unit intervals on the x and y axes to show the length scale.

$3\sin(0) = 0$. $\theta = 0$

$3\sin(\pi) = 0$. $\theta = \pi/2$

$3\sin(\pi/2) = 3$ $\theta = \pi/4$.



Question 10. (***11 points***) Find the area of the polar curve $r = 3\sin(2\theta)$ in the first quadrant.

$$A = \frac{1}{2}\int_0^{\pi/2} r^2\,d\theta = \frac{1}{2}\int_0^{\pi/2} 9\sin^2(4\theta)\,d\theta$$

$$= \frac{9}{2}\int_0^{\pi/2} \frac{1-\cos(8\theta)}{2}\,d\theta$$

$$= \frac{9}{4}\left[\theta - \frac{1}{8}\sin(8\theta)\right]_0^{\pi/2}$$

$$= \frac{9}{4}\left[\pi/2 - \frac{1}{8}\sin(4\pi)\right] = \frac{9\pi}{8}.$$

Question 11. Given vectors $\boldsymbol{u} = \langle 2,5 \rangle$ and $\boldsymbol{v} = \langle 3,4 \rangle$:

a. (**3 points**) Find $2\boldsymbol{u} - \boldsymbol{v}$,

$$2\langle 2,5\rangle - \langle 3,4\rangle = \langle 4,10\rangle - \langle 3,4\rangle$$

$$= \langle 1,6\rangle.$$

b. (**3 points**) Find $|\boldsymbol{u}|$,

$$|u| = \sqrt{4+25} = \sqrt{29}.$$

c. (**3 points**) Find a vector of length 2 in the direction of direction of $\boldsymbol{v}$.

$$\frac{2}{\sqrt{29}} \cdot \langle 2,5\rangle$$

d. (**2 points**) Express $\boldsymbol{u}$ as a combination of the unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$.

$$u = 2i + 5j.$$

Question 12

a. (***4 points***) Define the dot product $\boldsymbol{u} \cdot \boldsymbol{v}$ of the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in terms of their magnitudes and the angle θ between them.

$$u \cdot v = uv \cdot \cos\theta .$$

b. (***3 points***) Compute $\langle 1,2,3\rangle \cdot \langle 3,-2,1\rangle$.

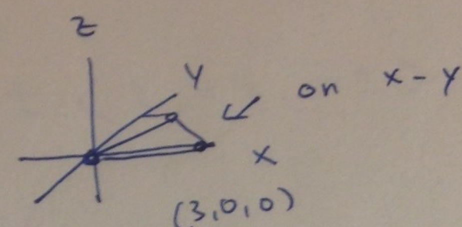
$$3 + (-4) + 3 = 2 .$$

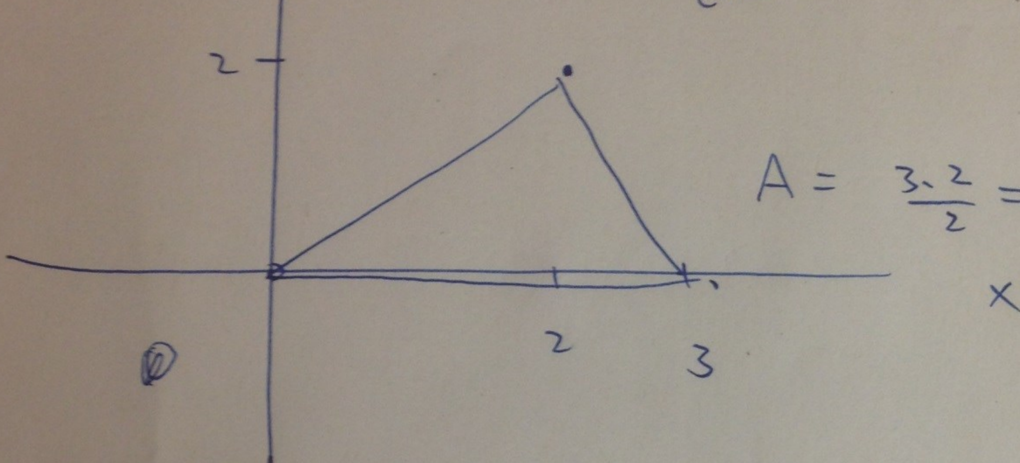
c. (***4 points***) Give the scalar projection of $\langle 1,2,3\rangle$ onto $\langle 3,-2,1\rangle$.

$$\frac{\langle 1,2,3\rangle \cdot \langle 3,-2,1\rangle}{|\langle 3,-2,1\rangle|}$$

$$= \frac{2}{\sqrt{9+4+1}} = \frac{2}{\sqrt{14}}$$

Question 13. (***10 points***) Find the area of the triangle whose vertices are the points $O(0,0,0)$, $P(3,0,0)$ and $Q(2,2,0)$.





$A = \frac{3 \cdot 2}{2} = 3.$

Question 14. (***11 points***) Find the equation of the line passing through the points $P(1,2,3)$ and $Q(3,-2,1)$.

$\langle 1,2,3 \rangle + t \langle 2,-4,-2 \rangle$.