Written Homework 4 Solutions

§ **8.5 #40**: Note

$$\left(\frac{1}{2}\right)^2 + \left(\frac{2}{3}\right)^3 + \left(\frac{3}{4}\right)^4 + \cdots \quad = \quad \sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n+1}.$$

To show this sum diverges, we will show that

$$\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{n+1} \neq 0.$$

To find this limit, we let

$$y = \left(\frac{n}{n+1}\right)^{n+1}.$$

Note we want to show that $\lim_{n\to\infty} y \neq 0$. Taking the natural log on both sides gives

$$\ln y = (n+1)\ln\left(\frac{n}{n+1}\right) = \frac{\ln\left(\frac{n}{n+1}\right)}{\frac{1}{n+1}}.$$

Then we take the limit on both sides to get

$$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} \frac{\ln\left(\frac{n}{n+1}\right)}{\frac{1}{n+1}}.$$

The right-hand limit is in the form $\frac{0}{0}$, so we can use L'Hospital's Rule, and we get

$$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} -\frac{n+1}{n} = -1.$$

So

$$\ln\left(\lim_{n\to\infty} y\right) = -1.$$

It follows that

$$\lim_{n\to\infty} y = e^{-1} = \frac{1}{e} \neq 0.$$

So the series diverges by the Divergence Test.

§ **8.5 #42**: Let

$$a_k = \left(\frac{k^2}{2k^2+1}\right)^k.$$

Note

$$\lim_{k\to\infty} \sqrt[k]{a_k} = \lim_{k\to\infty} \frac{k^2}{2k^2+1} = \frac{1}{2} < 1.$$

So the series converges by the root test.

§ **8.5 #44**: Note

$$\sum_{k=1}^{\infty} \frac{\sin^2(k)}{k^2} \quad \leq \quad \sum_{k=1}^{\infty} \frac{1}{k^2}$$

which is a convergent p-series. So the series converges by the comparison test.

§ **8.5 #46**:

Method 1: We use the ratio test. Let

$$a_k = \frac{2^k}{e^k - 1}.$$

Note

$$\begin{aligned} \lim_{k\to\infty} \frac{a_{k+1}}{a_k} &= \lim_{k\to\infty} \frac{2^{k+1}}{e^{k+1}-1} \cdot \frac{e^k - 1}{2^k} \\ &= \lim_{k\to\infty} \frac{2(e^k - 1)}{e^{k+1} - 1} \\ &= \lim_{k\to\infty} \frac{2(e^k - 1)}{e(e^k) - 1} \\ &= \frac{2}{e} < 1. \end{aligned}$$

So the series converges by the ratio test.

Method 2: Let

$$a_k = \frac{2^k}{e^k - 1}, \quad b_k = \frac{2^k}{e^k}.$$

Note

$$\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \left(\frac{2}{e}\right)^k$$

is a convergent geometric series, since $\frac{2}{e} < 1$. Next, note

$$L = \lim_{k\to\infty} \frac{a_k}{b_k} = 1.$$

Since $0 < L < \infty$, by the limit comparison test, either both series converge or both diverge. Since $\sum b_k$ converges, so does $\sum a_k$.

§ **8.5 #52**: Again, we use the ratio test. Let

$$a_k = \frac{(k!)^3}{(3k)!}.$$

Note

$$\begin{aligned}
\lim_{k\to\infty} \frac{a_{k+1}}{a_k} &= \lim_{k\to\infty} \frac{((k+1)!)^3}{(3(k+1))!} \cdot \frac{(3k)!}{(k!)^3} \\
&= \lim_{k\to\infty} \frac{(k+1)^3}{(3k+3)(3k+2)(3k+1)} \\
&= \frac{1}{27} < 1.
\end{aligned}$$

So the series converges by the ratio test.

§ **8.6 #12**: We use the Alternating Series Test. Let

$$a_k = \frac{1}{\sqrt{k}}.$$

Note

$$\begin{aligned}
&(1) \quad a_{k+1} = \frac{1}{\sqrt{k+1}} < \frac{1}{\sqrt{k}} = a_k, \quad \text{and} \\
&(2) \quad \lim_{k\to\infty} a_k = 0.
\end{aligned}$$

So, by the Alternating Series Test, the series converges.