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Section: - Solution -

Practice with Series Lab

Directions: In this lab you will be determining the convergence or divergence of a variety of series of the form $\sum_{n=1}^{\infty} a_n$. There are 3 parts to each question as follows:

A.) Apply the divergence test.

B.) Prediction: Based on the results of the divergence test and the form of the general term a_n predict **and then implement**, which test you would use to determine convergence or divergence. Be sure to show your work.

C.) Reflection: If your prediction is correct indicate which tip you used (or name your own strategy if it works) to determine your prediction, and what aspects of the general term a_n helped you in your prediction. Else, if your prediction failed try another until you find a test that works, then describe what aspect of the series might help inform another student to choose a test that does work for the series. Be sure to show your work for the test that does work.

$a_n = \dfrac{(2n-1)!}{(2n+1)!}$

1. A.) Divergence Test:

$$\lim_{n\to\infty} \frac{(2n-1)!}{(2n+1)!} = \lim_{n\to\infty} \frac{1}{2n(2n+1)} = 0.$$ Div. test fails.

B.) Predicted Test: Ratio test

Work:

$$\frac{a_{n+1}}{a_n} = \frac{(2(n+1)-1)!}{(2(n+1)+1)!} \cdot \frac{(2n+1)!}{(2n-1)!} = \frac{1}{(2n+3)(2n+2)} \cdot (2n+1) \cdot 2n = 1$$

C.)Reflection (Including work if predicted test failed):

Predicted test fails.

$$a_n = \frac{1}{2n(2n+1)} \leq \frac{1}{4n^2}$$

Thus, $\sum a_n \leq \sum \frac{1}{4n^2} \Rightarrow \sum a_n$ converges by p-series test.

2. $a_n = \frac{\sin(\frac{2}{n})}{2^n}$

A.) Divergence Test:

$$\lim_{n\to\infty} \frac{\sin(2/n)}{2^n} = 0$$ div. test fails.

B.) Predicted Test: Root test

Work:

$$\sqrt[n]{a_n} = \frac{\sqrt[n]{\sin(2/n)}}{2^{\infty}}$$ Since $\sqrt[n]{\sin(2/n)} < 1$,

$$\lim_{n\to\infty} \sqrt[n]{a_n} < 1$$

C.)Reflection (Including work if predicted test failed):

Root test works and the series conv.

3. $a_n = \frac{(-2)^{2n}}{n^n}$

A.) Divergence Test:

$$\lim_{n\to\infty} \frac{(-2)^{2n}}{n^n} = \lim_{n\to\infty} \left(\frac{4}{n}\right)^n = 0$$ div. test fails.

B.) Predicted Test: root test

Work:

$$\sqrt[n]{a_n} = \frac{(-2)^2}{n} = \frac{4}{n}. \qquad \lim_{n\to\infty} \sqrt[n]{a_n} < 1.$$

C.)Reflection (Including work if predicted test failed):

root test works and the series conv.

$$\lim a_n = \lim \frac{n!}{e^{n^2}} \overset{L.H.}{=} \lim \frac{O(n^{n-1})}{e^{n^2}\cdot 2n} \overset{L.H}{=} \lim \frac{O(n^{n-2})}{O(e^{n^2}\cdot n^2)} \cdots$$

4. $a_n = \frac{n!}{e^{n^2}}$

A.) Divergence Test:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n!}{e^{n^2}} = 0 \quad \text{since} \quad e^{n^2} = (e^n)^n \text{ and } e^n > n \text{ so}$$

$$e^{n^2} > n^n. \text{ So } \lim_{n\to\infty} a_n < \lim_{n\to\infty} \frac{n!}{n^n} < \lim \frac{1}{n}$$

B.) Predicted Test: ratio test

Work:

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{e^{(n+1)^2}} \cdot \frac{e^{n^2}}{n!} = \frac{(n+1)e^{n^2}}{e^{n^2}e^{2n}\cdot e} = \frac{n+1}{e^{2n+1}}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0 \text{ by L'Höspital}$$

C.)Reflection (Including work if predicted test failed):

ratio test works and the series conv.

5. $a_n = \dfrac{5^{n^2}}{4^n - 1}$

A.) Divergence Test:

$$\lim_{n \to \infty} \frac{5^{n^2}}{4^n - 1} = \lim_{n \to \infty} \left(\frac{5^n}{4}\right)^n = \infty$$

B.) Predicted Test: div. test.
Work:

$\lim a_n \neq 0$

C.)Reflection (Including work if predicted test failed):

div. test works and the series conv.

6.

$$\sum_{k=1}^{\infty} \frac{k^{\frac{53}{54}}}{520k^2 + 173k - 80000000000001}$$

A.) Divergence Test:

$\lim_{k \to \infty} a_k = 0$ div. test fails

B.) Predicted Test: p-series test

Work:

$a_k < \frac{k^{53/54}}{520k^2+172k}$ for sufficiently large enough k, say $k \geq N$.

Then $\sum_N^{\infty} a_k \leq \sum_N^{\infty} \frac{k^{53/54}}{520k^2+172k} \leq \sum_N^{\infty} \frac{k^{53/54}}{k^2} = \sum_N^{\infty} \frac{1}{k^{55/54}}$

C.)Reflection (Including work if predicted test failed):

By p-series test, the series conv.

7. $a_n = \frac{4^{n+2}}{7^{n-1}}$

A.) Divergence Test:

$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{4^2 \cdot 4^n}{7^{-1} \cdot 7^n} = \lim_{n \to \infty} 4^2 \cdot 7 \cdot \left(\frac{4}{7}\right)^n = 0.$ div. test fails.

B.) Predicted Test: root test

Work:

$\sqrt[n]{a_n} = \sqrt[n]{4^2 \cdot 7} \cdot \sqrt[n]{\left(\frac{4}{7}\right)^n} = \frac{4}{7} \cdot \sqrt[n]{4^2 \cdot 7}$

$\lim_{n \to \infty} \sqrt[n]{a_n} = 0 < 1$

C.)Reflection (Including work if predicted test failed):

root test works and the series conv.

Challenge: $a_n = \dfrac{\ln\left(1+\frac{1}{n}\right)}{n^2+n}$ $= \dfrac{1}{n(n+1)} \ln\left(\dfrac{n+1}{n}\right)$

Hint: Try reducing a_n first.

A.) Divergence Test:

$\lim\limits_{n\to\infty} a_n < \lim\limits_{n\to\infty} \ln\left(\dfrac{n+1}{n}\right) = 0$. div. test fails.

B.) Predicted Test: ratio test

Work:

$$\frac{a_{n+1}}{a_n} = \frac{1}{\cancel{(n+1)}(n+2)} \cdot \ln\left(\frac{n+2}{n+1}\right) \cdot \cancel{(n+1)} \cdot n \left[\ln\left(\frac{n+1}{n}\right)\right]^{-1}$$

$$= \frac{n \cdot \ln\left(\frac{n+2}{n+1}\right)}{(n+2)\ln\left(\frac{n+1}{n}\right)} = \frac{n \cdot \ln\left(1+\frac{1}{n+1}\right)}{(n+2)\cdot\ln\left(1+\frac{1}{n}\right)} \longrightarrow 1$$

C.)Reflection (Including work if predicted test failed):

Ratio test fails.

Note $\left|\ln\left(1+\frac{1}{n}\right)\right| < |\ln(2)| < 1$. Thus,

$$|a_n| < \frac{1}{n^2+n} < \frac{1}{n^2}.$$

$\sum a_n \leq \sum |a_n| < \sum \frac{1}{n^2}$. Thus, the series conv. by p-series test.