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Section: - Solution -

Sequences and Series Lab

Introduction
A *sequence* is an ordered list of numbers; the numbers in this ordered list are called *elements* or terms. A *series* is a sum of the terms of a sequence, which we call the *sum*. For instance, $\{1, 2, 3, ...\}$ is an example of an *infinite* sequence whose terms start at 1 and go up by 1 each term without end. On the other hand the $\{2, 4, 6, 8\}$ is an example of a *finite* sequence since it has finitely many terms. Correspondingly we can find the sum of these sequences as

$$1 + 2 + 3 + 4 + 5 + ... = \sum_{n=1}^{\infty} n$$

and

$$2 + 4 + 6 + 8 = \sum_{n=1}^{4} 2n$$

In general it is customary to denote an infinite sequence or series of terms as $\boxed{\{a_i\}_{i=1}^{\infty} = \{a_1, a_2, a_3, a_4, ...\}}$ and $\boxed{\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + ...}$ while denoting a finite sequence or series of terms as $\{a_i\}_{i=1}^{n} = \{a_1, a_2, a_3, ..., a_n\}$ and $\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + ... + a_n$. In the above finite case we had $n = 4$ terms, which were $a_1 = 2$, $a_2 = 4$, $a_3 = 6$ and $a_4 = 8$. We were able to express any term as a function of its place in the sequence, namely we said $a_k = 2k$. Notice the relationship here between the index of a_k and the term $2k$. Ideally we would like to be able to write a sequence or series in its most compact form since determining properties like convergence are much easier to find if we know the form of the general term or a_n. However, it is not always guaranteed that we can write the general form as a function of its index or place in the sequence as we will soon see.

Exercises:

Directions I_1 and I_2: Find the relationship between the terms and their place value or index then write the sequence or series in its most compact form, e.g. In the above example the infinite sequence could be written compactly as $\{n\}_{n=1}^{\infty}$ and the finite sequence as $\{2n\}_{n=1}^{4}$. Note that we have already written their corresponding series compactly. If no relationship exists then state as much.

I_1 (a) $3, 6, 9, 12, ...$

$\{3n\}_{n=1}^{\infty}$

(b) $6, 9, 12, 15, 18, ..., 33$

$\{6+3n\}_{n=0}^{9}$

(c) $2, -4, 6, -8, 10, ..., -20$ (We call a sequence of this type an alternating sequence since the signs *alternate* between terms).

$\{(-1)^n(2+2n)\}_{n=0}^{9}$

I_2 (a) $1 + 3 + 5 + 7 + 9 + \ldots$

$$\sum_{n=0}^{\infty} (1+2n)$$

(b) $7 + 9 + 11 + 13 + \ldots + 23$

$$\sum_{n=0}^{8} (7+2n)$$

(c) $\frac{5}{6+3} + \frac{5}{7+3} + \frac{5}{8+3} + \ldots + \frac{5}{31+3}$

$$\sum_{n=0}^{25} \frac{5}{(6+n)+3}$$

Directions 1-3: Sometimes we wish to rewrite a sequence or series so that it starts with a particular index, which is not necessarily 1. Express/re-express each sequence in compact form so that it starts at the indicated index. Hint: It may help to write out the first couple terms to see what is going on.

1. $\{15, 17, 19, 21, 23\} = \{2n+3\}_{n=6}^{10}$ starting at $n = 1$ instead of $n = 6$ (the original starting index).

$$\{2n+13\}_{n=1}^{5}$$

2. $\{a_n\}_{n=1}^{\infty}$ where $a_n = n^2$ starting at $n = 4$ instead of $n = 1$.

$$\{(n-3)^2\}_{n=4}^{\infty}$$

Geometric Sequences and Series:

A geometric sequence, a_k, is any sequence such that the ratio of any two consecutive terms is constant, i.e. $a_{k+1}/a_k = r$ for some real number r. The sum of the first n terms of a geometric sequence is called n^{th} partial sum of the given geometric sequence, usually denoted as $S_n = \sum_{n=0}^{n-1} a_k$. If we take an infinite sum of a geometric sequence, i.e. n goes to ∞, then we call the sum geometric series, usually denoted as $S = \sum_{n=0}^{\infty} a_k$.
For example, $\{1, 2, 4, 8, 16, \cdots\}$ is a geometric sequence with $a_0 = 1$, constant ratio $r = 2$. Then, the sequence of n^{th} partial sums is $\{1, 3, 7, 15, 31, \cdots\}$.

3. Express the n^{th} term of a geometric series in terms of its ratio r and the first value, a_0. (hint. Note for $\{1, 2, 4, 8, 16, \cdots\}$, $a_n = 2^{n-1}$ where $r = 2$ and $a_0 = 1$.)

$$a_n = a_1 \cdot r^{n-1}$$

4. Express the following sequence in terms of the ratio, r, and the first value, a_0. Then compute the indicated sums (you may use a calculator to evaluate the sums):

(a) $\{a_k\} = \{1, 3, 9, 27, \cdots\}$. $a_n = 1 \cdot 3^{n-1}$

$$\sum_{k=2}^{4} a_k = (1 \cdot 3^{(2-1)}) + (1 \cdot 3^{(3-1)}) + (1 \cdot 3^{(4-1)})$$
$$= (1 \cdot 3) + (1 \cdot 3^2) + (1 \cdot 3^3)$$
$$= 3 + 9 + 27 = \boxed{39}$$

(b) $\{b_k\} = \{3, 6, 12, \cdots\}$. $b_n = 3 \cdot 2^{n-1}$

$$\sum_{k=0}^{4} b_k = (3 \cdot 2^{0-1}) + (3 \cdot 2^{1-1}) + (3 \cdot 2^{2-1}) + (3 \cdot 2^{3-1}) + (3 \cdot 2^{4-1})$$
$$= (3 \cdot 2^{-1}) + (3 \cdot 2^0) + (3 \cdot 2^1) + (3 \cdot 2^2) + (3 \cdot 2^3)$$
$$= (1.5) + (3) + (6) + (12) + (24)$$
$$= \boxed{46.5}$$

5. Although we can evaluate each terms in a geometric sequence and add them together in order to evaluate the n^{th} partial sum, it will get challenging as n increases. Therefore, we want to find a general formula that will evaluate n^{th} partial sum of a geometric sequence. First, assume $r \neq 1$ (the reason we don't let $r = 1$ will become clear later). Then the n^{th} partial sum of a geometric series is $S_n = \sum_{n=0}^{n-1} a_k = a_0 + a_1 + \cdots + a_{n-1}$.

(a) First, we want to evaluate rS_n.

$$rS_n = r\sum_{n=0}^{n-1} a_k = ra_0 + ra_1 + \cdots + ra_{n-1} = \boxed{a - ar^n}$$

(b) Now, we subtract rS_n from S_n

$$S_n - rS_n = (1-r)S_n = \sum_{n=0}^{n-1} a_k - r\sum_{n=0}^{n-1} a_k = (1-r)a_0 + (1-r)a_1 + \cdots + (1-r)a_{n-1}$$

(c) Now, we divde both sides by $1-r$. $\frac{(1-r)S_N}{1-r} = \frac{(1-r)a_0}{(1-r)} + \frac{(1-r)a_1}{(1-r)} + \cdots + \frac{(1-r)a_{n-1}}{(1-r)}$

(Note this is why we assume $r \neq 1$. We cannot divide by 0.)

$$= \frac{a - ar^n}{(1-r)} = \boxed{\frac{a(1-r^n)}{(1-r)}}$$

(d) Therefore, the n^{th} partial sum of a given geometric series in terms of r and a_0 is:

$$S_n = \sum_{n=0}^{n-1} a_k = a_0 + a_1 + \cdots + a_{n-1} = \frac{a(1-r^n)}{1-r}$$

(e) Now we deal with $r = 1$. Note $a_{k+1} = ra_k = a_k$ in this case, i.e. $a_k = a_0$ for all k. Therefore,

$$S_n = \sum_{n=0}^{n-1} a_k = a_0 + a_0 + \cdots + a_0 = \frac{a(1-\cancel{r^n})}{1-r}$$ (hint. how many a_0 do we have?)

0

$\rightarrow \frac{a}{1-r}$ if $r^n = 0$

0 if $r^n = 1$ because $\frac{a(1-1)}{1-r} = 0$

$$S_n = a + \cancel{ar} + \cancel{ar^2} + \cdots + ar^{n-1}$$
$$- \quad rS_n = \cancel{ar} + \cancel{ar^2} + ar^3 + \cdots + ar^n$$
$$S_n - rS_n = a - ar^n \Longrightarrow \boxed{S_n(1-r) = a(1-r^n)}$$

6. Our last goal is to find a general expression for an infinite sum. In order to get a "nice" answer, we shall assume $|r| < 1$ ("nice" answer will be introduced as converging answer later on). An infinite sum just means we take the partial sum and let n goes to ∞, i.e. take a limit of the partial sum as n goes to ∞. Therefore,

$$S = \sum_{n=0}^{\infty} a_k = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{(1-r)} \Longrightarrow \lim_{n\to\infty} r^n = \begin{cases} 0 & -1<r<1 \\ \text{diverge} & |r|>1 \end{cases}$$

Now, using the fact that $\lim_{n\to\infty} r^n = 0$ for all r such that $|r| < 1$ (this is what gives us the "nice" answer), we have

$$S = \frac{a}{1-r}$$

Thus, the theorem is as follows: The geometric series: $a + ar + ar^2 + \cdots$ is $\boxed{\text{convergent}}$ if $|r| < 1$ and its sum is $\frac{a}{1-r}$. Otherwise, if $|r| > 1$, then the geometric series $\boxed{\text{diverges}}$.