Final Exam Study Guide

MATH 140 Lab: Section 1

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Note: This study guide contains my practice questions that I think will be useful for preparing you for the final exam in Calculus for Life Scientists.

Question 1: Evaluate the following limit:

$$\lim_{k\to\infty}(\ln k + e^k) = \ln(\infty) + e^{\infty} =$$

$$= \infty + \infty$$

$$= \boxed{\infty}$$

Question 2: Evaluate the following limit:

$$\lim_{n\to 1}\left(\frac{\tan(n-1)}{(n-1)}\right)$$

let $u = n - 1$ $\Rightarrow$ If $n = 1$, then $\boxed{u = 0}$

then, we obtain $\lim_{u\to 0} \frac{\tan(u)}{u} = 1$

Notes:

$\lim_{x\to 0} \frac{\tan(x)}{x} = 1$

$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$

Question 3: Evaluate the following limit:

$$\lim_{x\to-\infty}\left(\frac{(2x^2+1)^2-x^4+x+1}{1-x-2x^2}\right)$$

leading terms

$$=\lim_{x\to-\infty}\frac{4x^4-x^4}{-2x^2}=\lim_{x\to-\infty}\frac{3x^{\cancel{4}\,2}}{-2\cancel{x^2}}=\lim_{x\to-\infty}\frac{3x^2}{-2}=$$

$$=\frac{3(-\infty)^2}{-2}=\boxed{-\infty}$$

Question 4: Evaluate the following limit:

$$\lim_{z\to1}\left(\frac{\tan(z-1)}{z^3-1}\right)$$

$$=\lim_{z\to1}\left(\frac{\tan(z-1)}{(z-1)(z^2+z+1)}\right)\Rightarrow$$

let $u=z-1$ then if $z=1$, then we obtain $\boxed{u=1-1=0}$

$$=\lim_{z\to1}\left(\frac{\tan(z-1)}{(z-1)}\right)\cdot\lim_{z\to1}\left(\frac{1}{z^2+z+1}\right)$$

$$=\lim_{u\to1}\left(\frac{\tan(u)}{u}\right)\cdot\left(\frac{1}{1+1+1}\right)=1\cdot\left(\frac{1}{3}\right)=\boxed{\frac{1}{3}}$$

Question 5: Find y' without simplifying your final answer:

$$y=\sqrt{\frac{2x^2e^x(x+1)}{x^2+1}}=\left(\frac{2x^2e^x(x+1)}{x^2+1}\right)^{1/2}$$

Take ln of both sides, we obtain:

$$\ln y=\ln\left(\frac{2x^2e^x(x+1)}{x^2+1}\right)^{1/2}\Rightarrow\ln y=\frac{1}{2}\left[\ln\left(\frac{2x^2e^x(x+1)}{(x^2+1)}\right)\right]$$

By using the natural log. properties, we obtain:

$$\ln y=\frac{1}{2}\left[\ln(2x^2)+\ln(e^x)+\ln(x+1)-\ln(x^2+1)\right]$$

Now, take the derivative of both sides, we obtain

$$y'/y=\frac{1}{2}\left[\frac{4x}{2x^2}+\frac{e^x}{e^x}+\frac{1}{x+1}-\frac{2x}{x^2+1}\right]\Rightarrow\boxed{y'=\frac{1}{2}\left[\frac{2}{x}+1+\frac{1}{x+1}-\frac{2x}{x^2+1}\right]\cdot\left(\sqrt{\frac{2x^2e^x(x+1)}{x^2+1}}\right)}$$

Question 6: Find y' without simplifying your final answer:

$$y = (\sin(x))^x$$

Take ln for both sides, we obtain:

$$\ln y = \ln(\sin(x))^x$$

$$\Rightarrow \ln y = x \cdot \ln(\sin(x))$$

Now, take the derivative of both sides, we obtain

$$\frac{y'}{y} = x \cdot \frac{\cos(x)}{\sin(x)} + 1 \cdot \ln(\sin(x)) \Rightarrow y' = \left[x \cdot \frac{\cos(x)}{\sin(x)} + \ln(\sin(x))\right] \cdot y$$

$$\Rightarrow \boxed{y' = \left[x \cdot \frac{\cos(x)}{\sin(x)} + \ln(\sin(x))\right] \cdot (\sin(x))^x}$$

Question 7: Evaluate the following limit:

$$\lim_{x\to 0^+}\left(\frac{1}{x} - \frac{1}{e^x - 1}\right)$$

$$\boxed{\infty - \infty = \frac{0}{0}}$$

$$= \lim_{x\to 0^+}\left[\frac{e^x - 1 - x}{x(e^x - 1)}\right]$$

L'Hôpital Rule

$$= \lim_{x\to 0^+}\left[\frac{e^x - 1 - x}{xe^x - x}\right] = \lim_{x\to 0^+}\left(\frac{e^x - 1}{xe^x + e^x - 1}\right) = \lim_{x\to 0^+}\frac{e^x}{xe^x + e^x + e^x}$$

$$\boxed{\frac{0}{0}} \qquad \boxed{\frac{0}{0}}$$

$$= \frac{e^0}{0 + e^0 + e^0} = \frac{1}{1+1} = \boxed{\frac{1}{2}}$$

Question 8: Evaluate the following limit:

$$\lim_{x\to 0}\left(\frac{x^3}{x - \tan(x)}\right) \longrightarrow \boxed{\frac{0}{0}}$$

L'Hôpital Rule

$$= \lim_{x\to 0}\frac{3x^2}{1 - \sec^2 x} = \lim_{x\to 0}\frac{6x}{-2\sec^2 x \cdot \tan x} =$$

$$\boxed{\frac{0}{0}} \qquad \boxed{\frac{0}{0}}$$

$$= \lim_{x\to 0}\frac{-6}{2\sec^2(x)\cdot\sec^2(x) + \tan(x)\cdot 4\sec^2(x)\cdot\tan(x)} = \frac{-6}{2+0} = \boxed{-3}$$

Question 9: Use $\boxed{\text{L'Hôpital's rule}}$ to find the following limit:

$$\frac{1-1+\ln(1)}{1+\cos(\pi)} = \boxed{\frac{0}{0}}$$

$$\lim_{x\to 1}\left(\frac{1-x+\ln x}{1+\cos(\pi x)}\right)$$

L'H

$$= \lim_{x\to 1}\left(\frac{-1+\frac{1}{x}}{-\pi\sin(\pi x)}\right) = \frac{-1+\frac{1}{1}}{-\pi\sin(\pi)} = \boxed{\frac{0}{0}}$$

L'H

$$= \lim_{x\to 1}\left(\frac{\frac{-1}{x^2}}{-\pi^2\cos(\pi x)}\right) = \frac{\frac{-1}{1}}{-\pi^2\cos(\pi)} = \frac{-1}{-\pi^2(-1)} = \boxed{\frac{-1}{\pi^2}}$$

Question 10: Find $\frac{dy}{dx}$:

$$y = \sin\left(x+\sqrt{x^2+5}\right)$$

$$\frac{dy}{dx} = \cos\left(x+\sqrt{x^2+5}\right)\cdot\left(1+\frac{x}{\sqrt{x^2+5}}\right)$$

Question 11: Find $\frac{d^2y}{dx^2}$:

$$xy + x^2 = y^3$$

$$\boxed{xy'} + y + 2x = \boxed{3y^2y'}$$

$$y+2x = 3y^2y' - xy' \Rightarrow y+2x = y'(3y^2-x)$$

$$\Rightarrow \boxed{y' = \frac{y+2x}{3y^2-x}} \Rightarrow \text{So, } \boxed{y'' = \frac{(3y^2-x)\cdot(y'+2)-(6yy'-1)\cdot}{(3y^2-x)^2}}$$

$(y+2x)$

$$\text{So, } \boxed{\frac{d^2y}{dx^2} = \frac{(3y^2-x)(y'+2)-(6yy'-1)(y+2x)}{(3y^2-x)^2}}$$

Question 12: Find the equation of the tangent line at the point $(0,1)$ to the following curve:

x_1, y_1

$$x^2y + 7y = 3e^x + 4$$

$y - y_1 = m(x - x_1)$

To find m, we need to find y' as follows:

$2xy + \boxed{x^2y'} + \boxed{7y'} = 3e^x + 0 \Rightarrow y'(x^2+7) = 3e^x - 2xy$

$y' = \dfrac{3e^x - 2xy}{x^2+7} \Rightarrow y'\Big|_{(0,1)} = \dfrac{3e^0 - 2(0)(1)}{(0)^2+7} = \dfrac{3-0}{7} = \boxed{\dfrac{3}{7}}$

So, $y - 1 = \frac{3}{7}(x - 0)$

$\boxed{y = \frac{3}{7}x + 1}$

Question 13: Evaluate the following limit:

$$\lim_{x\to\infty}\left(\frac{x+3}{x}\right)^{2x}$$

$\lim\limits_{x\to\infty}\left(1+\frac{3}{x}\right)^{2x} = \lim\limits_{x\to\infty} e^{\ln(1+\frac{3}{x})^{2x}} = \lim\limits_{x\to\infty} e^{2x\ln(1+\frac{3}{x})} = \lim\limits_{x\to\infty} e^{\frac{\ln(1+3/x)}{(1/2x)}} = e^{\lim\limits_{x\to\infty}\frac{\ln(1+\frac{3}{x})}{(1/2x)}}$ $\boxed{\frac{\infty}{\infty}}$

$\overset{'H}{=} e^{\lim\limits_{x\to\infty}\frac{\frac{-3/x^2}{(1+\frac{3}{x})}}{\frac{-1}{2x^2}}} = e^{\lim\limits_{x\to\infty}\left[\frac{\frac{-3}{\cancel{x^2}}}{(1+3/x)}\cdot\frac{-2\cancel{x^2}}{1}\right]} = e^{\lim\limits_{x\to\infty}\left(\frac{6}{1+\frac{3}{x}}\right)}$

$= e^{\left(\frac{6}{1+0}\right)} = \boxed{e}$

Question 14: Given the following function:

$$f(x) = x(x-1)^3$$

Part a: Find the x and y intercepts for the graph of f.

x - intercept when $y = 0 \Rightarrow x(x-1)^3 = 0$

$\boxed{x = 1}$ or $\boxed{x = 1}$

y - intercept when $x = 0$

$\Rightarrow \boxed{y = 0}$

Part b: Find the intervals on which the function is increasing and decreasing and locate any local extrema.

Product Rule

$$f'(x) = x \cdot 3(x-1)^2 \cdot 1 + (x-1)^3 \cdot 1$$

$$\Rightarrow f'(x) = 3x(x-1)^2 + (x-1)^3$$

$$\Rightarrow f'(x) = (x-1)^2\left[3x + (x-1)\right] = 0$$

$$\boxed{x = 1} \qquad 3x + (x-1) = 0 \Rightarrow 3x + x - 1 = 0$$

$$4x - 1 = 0$$

$$4x = 1 \Rightarrow \boxed{x = \frac{1}{4}}$$

f': $-$ | $\frac{1}{4}$ | $+$ | 1 | $+$

$\searrow$ $\frac{1}{4}$ $\nearrow$ 1 $\nearrow$

at $x = \frac{1}{4} \Rightarrow$ local min

$(-\infty, \frac{1}{4})$ decreasing

$(\frac{1}{4}, \infty)$ increasing

Part c: Find the intervals on which the function is concave up and concave down and identify aby inflection points if there are any.

$$f''(x) = 3x \cdot 2(x-1) + (x-1)^2 \cdot 3 + 3(x-1)^2 \cdot 1 = 0$$

$$f''(x) = 6x(x-1) + 3(x-1)^2 + 3(x-1)^2$$

$$= 6x(x-1) + 6(x-1)^2$$

$$= 6x^2 - 6x + 6(x^2 - 2x + 1)$$

$$= 6x^2 - 6x + 6x^2 - 12x + 6$$

$$= \boxed{12x^2 - 18x + 6}$$

Inflection points

$\boxed{x = \frac{1}{2}}$ or $\boxed{x = 1}$

$+$ | $\frac{1}{2}$ | $-$ | 1 | $+$

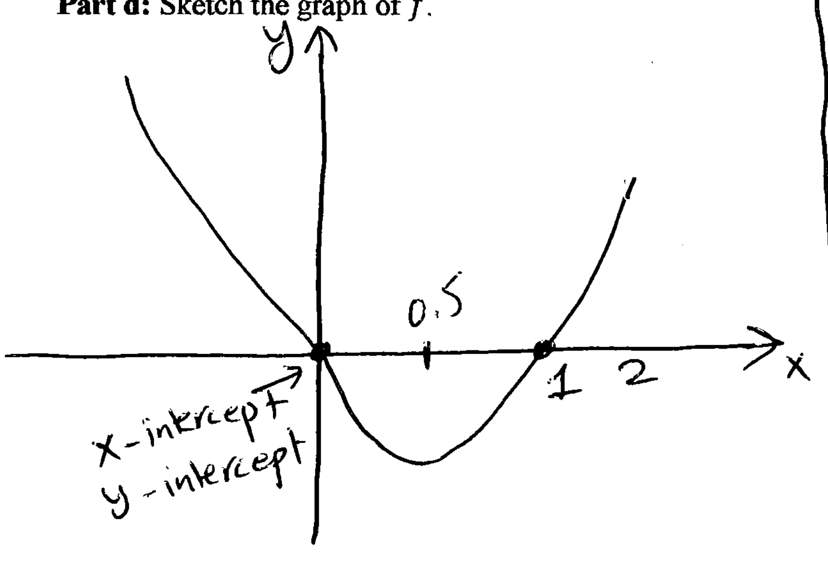
$\cup$ Concave up, $\cap$ Concave down, $\cup$ Concave up

$(-\infty, \frac{1}{2})$ Concave up

$(\frac{1}{2}, 1)$ Concave down

$(1, \infty)$ Concave up

Part d: Sketch the graph of f.



(f) $= \int x^3(x - 6\sqrt{x} + 9)\,dx = \int (x^4 - 6x^{7/2} + 9x^3)\,dx = \frac{x^5}{5} - 6\frac{x^{9/2}}{9/2} + \frac{9x^4}{4} + C$

$= \boxed{\frac{1}{5}x^5 - \frac{4}{3}x^{9/2} + \frac{9}{4}x^4 + C}$

Question 15: Find the indefinite integral for the following:

a. $\int \frac{(x+2)(x-1)}{x}\,dx$

b. $\int \frac{4x^3+2}{2x^4+4x+1}\,dx$

c. $\int (\sec x\tan(x) + \cos(3x) - 5)\,dx$

d. $\int \frac{x^4+3}{x}\,dx$

e. $\int \left(e^{3x} + \frac{1}{\sqrt[3]{x}}\right)dx$

f. $\int x^3(\sqrt{x} - 3)^2\,dx$

(e) $\int \left(e^{3x} + \frac{1}{\sqrt[3]{x}}\right)dx =$

$\int e^{3x}\,dx + \int x^{-1/3}\,dx =$

$= \frac{e^{3x}}{3} + \frac{x^{2/3}}{2/3} + C$

$= \boxed{\frac{1}{3}e^{3x} + \frac{3}{2}x^{2/3} + C}$

(a) $\int \frac{(x+2)(x-1)}{x}\,dx = \int \frac{(x^2+2x-x-2)}{x}\,dx = \int \frac{x^2+x-2}{x}\,dx =$

$= \int (x + 1 - 2x^{-1})\,dx = \boxed{\frac{x^2}{2} + x - 2\ln|x| + C}$

(b) $\int \frac{4x^3+2}{2x^4+4x+1}\,dx = \boxed{\frac{1}{2}\ln|2x^4+4x+1| + C}$

(c) $\int (\sec(x)\tan(x) + \cos(3x) - 5)\,dx = \int \sec(x)\tan(x)\,dx +$

$\int \cos(3x) - \int 5\,dx = \boxed{\sec(x) + \frac{\sin(3x)}{3} - 5x + C}$

(d) $\int \frac{x^4+3}{x}\,dx = \int \frac{x^4}{x}\,dx + \int \frac{3}{x}\,dx = \int x^3\,dx + \int 3x^{-1}\,dx =$

$= \boxed{\frac{x^4}{4} + 3\ln|x| + C}$

Question 16: Find the definite integral for the following:

$$\int_1^e \frac{(\ln(x))^2}{x}\,dx$$

$u = \ln x$

$\dfrac{du}{dx} = \dfrac{1}{x} \Rightarrow du = \dfrac{1}{x}\,dx$

$\displaystyle\int_1^e \frac{(\ln(x))^2}{x}\,dx = \int u^2\,du = \frac{u^3}{3} = \left.\frac{(\ln x)^3}{3}\right|_1^e =$

$= \dfrac{1}{3} - 0 = \boxed{\dfrac{1}{3}}$

Question 17: Find the following integral for the following:

$$\int \cos(\sqrt[3]{x})\,dx$$

Hint: Use substitution and integration by parts

let $w = \sqrt[3]{x} = x^{1/3}$

then, $w^3 = (x^{\frac{1}{3}})^3 = x$

Take the derivative of both sides, we obtain

$\boxed{3w^2\,dw} = dx$

So, $\displaystyle\int \cos(w)(3w^2\,dw) = 3\int w^2\cos(w)\,dw$

Integration by parts: let's use table method

/	$\int$
w^2	$\cos(w)$
$2w$	$\sin(w)$ (+)
2	$-\cos(w)$ (−)
0	$-\sin(w)$ (+)

$\Rightarrow = 3\left[w^2\sin(w) + 2w\cos(w) - 2\sin(w)\right] + C$

$= 3\left[(\sqrt[3]{x})^2\sin(\sqrt[3]{x}) + 2\sqrt[3]{x}\cos(\sqrt[3]{x}) - 2\sin(\sqrt[3]{x})\right] + C$

Question 18: Solve the following differential equation:

$$\frac{dy}{dx} = e^{3y+2x} = e^{3y} \cdot e^{2x}$$

$$\Rightarrow \frac{dy}{dx} \times \frac{\frac{1}{e^{2x}}}{\frac{1}{e^{3y}}} \Rightarrow \frac{1}{e^{3y}}\,dy = \frac{1}{e^{2x}}\,dx \Rightarrow \frac{1}{e^{3y}}\,dy - \frac{1}{e^{2x}}\,dx = 0$$

Take the integral of both sides, we obtain:

$$\int \frac{1}{e^{3y}}\,dy - \int \frac{1}{e^{2x}}\,dx = \int 0 \Rightarrow \int e^{-3y}\,dy - \int e^{-2x}\,dx = C$$

$$\Rightarrow \frac{e^{-3y}}{-3} - \frac{e^{-2x}}{-2} = C \Rightarrow -\frac{1}{3}e^{-3y} + \frac{1}{2}e^{-2x} = C$$ By taking ln of both sides

$$\Rightarrow -\frac{1}{3}\ln e^{-3y} + \frac{1}{2}\ln e^{-2x} = \ln C$$ $\boxed{\text{Assume } \ln C = C}$

$$\Rightarrow -\frac{1}{\cancel{3}}(\cancel{-3}y) + \frac{1}{\cancel{2}}(\cancel{-2}x) = C$$

$$\Downarrow$$

$$y - x = C$$

$$\boxed{y = x + C}$$

Question 19: Given the following function:

$$y(t) = \frac{a}{k}(1 - e^{-kt})$$

Assume that an antibiotic with half-life $T_{\frac{1}{2}} = 12$ ***hour*** is given to a patient intravenously at a rate of $a = 50\ mg/hour$.

Part a: Find the rate constant k. From the half-life equation: $\boxed{T_{1/2} = \frac{\ln 2}{K} = \text{Constant}}$ So, from this equation, we need to find K as follows:

$$\frac{12}{1} \times \frac{\ln 2}{K} \Rightarrow 12K = \ln 2 \Rightarrow \boxed{K = \frac{\ln 2}{12} \approx 0.0578}$$

Part b: Given that: $\frac{dy}{dt} = a - ky$. What is the steady state solution for the amount of drug delivered by infusion in $\frac{dy}{dt}$.

$$\frac{dy}{dt} = a - ky \Rightarrow \frac{dy}{dt} = a - ky = 0 \Rightarrow a - ky = 0$$

$$\Rightarrow \boxed{a = ky} \Rightarrow \boxed{y = \frac{a}{K}}$$

The steady-state solution is $y = \frac{a}{K} \Rightarrow \boxed{y(t) = \frac{a}{K}}$.

Question 20: A rectangle has its base on the x-axis and its upper two vertices on the parabola $y = 12 - x^2$. Find the largest area that the rectangle can have?

Step 1: $A = 2xy$

Step 2: $y = 12 - x^2$

Step 3: $A = 2x(12 - x^2)$ $[0, \sqrt{12}]$

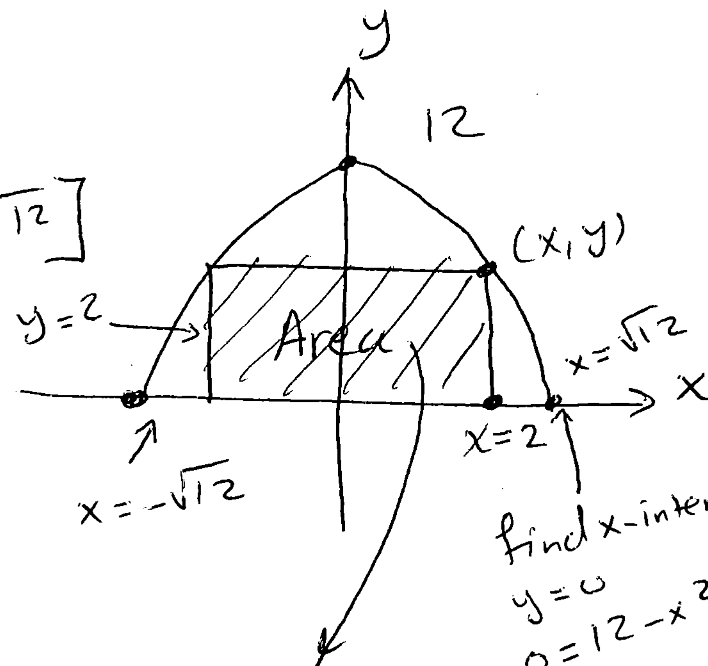
$\Rightarrow A = 24x - 2x^3$

Step 4: $A' = 24 - 6x^2 = 0$

$\Rightarrow 24 = 6x^2$

$\Rightarrow 4 = x^2$

$\Rightarrow \boxed{x = 2}$



find x-intercept

$y = 0$

$0 = 12 - x^2$

$x^2 = 12$

$\boxed{x = \sqrt{12}}$

$\text{Area} = 2xy$

$= 2(2)(12 - 4)$

$= 2(2)(8)$

$= \boxed{32}$

Good Luck in Final Exam

Mohammed Kaabar