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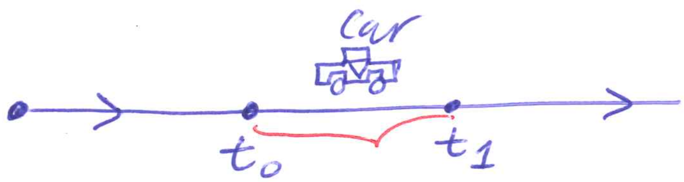
* Review about Average Speed & Instanteneous Velocity:



This car is moving from time t_0 to time t_1. Assume the speed of this car can be written as $m(t)$.

① Find Average Speed:

$$\text{Average Speed} = \boxed{\frac{m(t_1) - m(t_0)}{t_1 - t_0}}$$

② Find Instanteneous Velocity:

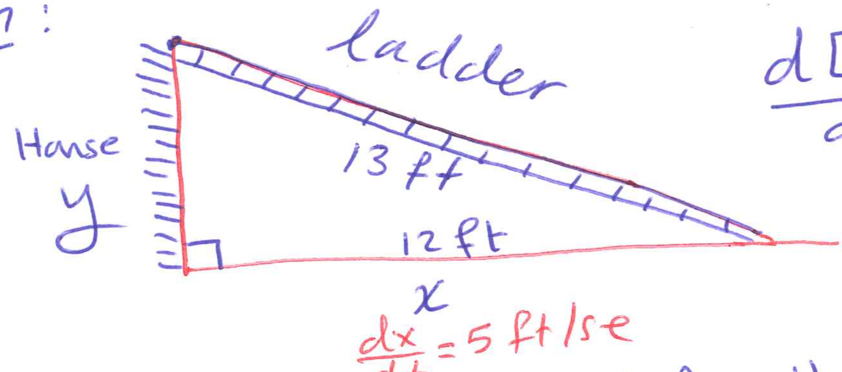
$$\text{Instanteneous Velocity} = \boxed{\lim_{t_1 \to t_0} \frac{m(t_1) - m(t_0)}{t_1 - t_0}}$$

Note: Average Speed gives you the average, while Instanteneous Velocity gives you the exact velocity.

$\Longrightarrow$

Ex 1 | A house has a ladder of $\boxed{13\text{ ft}}$ which is leaning against it when the house's base starts to slide away. By the time, the house base is $\boxed{12\text{ ft}}$ from the house, the base is moving at the rate of $\boxed{5\text{ ft/sec}}$.

ⓐ How fast is the top of ladder sliding down the wall? Solution:



$\frac{d\boxed{?}}{dt}$

① $\frac{dy}{dt} = ?$

② $x^2 + y^2 = (13)^2$

$\Rightarrow x^2 + y^2 = 169$

③ Take the derivative with respect to t for the above both sides:

$$2x\frac{dx}{dt} + 2y\boxed{\frac{dy}{dt}} = 0 \rightarrow \text{We need to find it.}$$

$x \rightarrow 12$, $\frac{dx}{dt} \rightarrow 5$

$y^2 = 169 - 144 = 25$

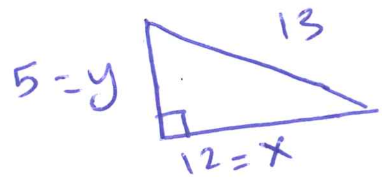
$y = \sqrt{25} = 5$

$$\Rightarrow 2(12)(5) + 2(5)\frac{dy}{dt} = 0$$

$$\Rightarrow 120 + 10\frac{dy}{dt} = 0 \Rightarrow 120 = -10\frac{dy}{dt} \Rightarrow \boxed{\frac{dy}{dt} = \frac{-120}{10} = -12\text{ ft/sec}}$$

(b) At what rate is the area of triangle formed by the ladder, wall, and ground changing them?

Solution:



(1.) $\frac{dA}{dt} = ?$

(2.) Area $= \frac{1}{2} \times$ base $\times$ height

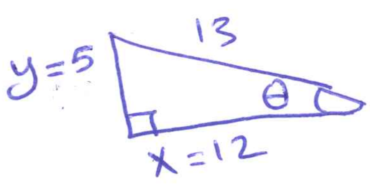
$$\Rightarrow \boxed{A = \frac{1}{2} x y}$$

(3.) Take the derivative for the above both sides:

$$\frac{dA}{dt} = \frac{1}{2}\left[\underset{12}{x} \cdot \underset{-12}{\frac{dy}{dt}} + \underset{5}{y} \cdot \underset{5}{\frac{dx}{dt}}\right]$$

$$\frac{dA}{dt} = \frac{1}{2}\left(12 \cdot (-12) + (5) \cdot (5)\right) = \frac{-119}{2} = \boxed{-59.5 \text{ ft}^2/\text{sec}}$$

(c) At what rate is the angle θ between the ladder and ground changing them?



(1.) $\frac{d\theta}{dt} = ?$

(2.) $\sin\theta = \frac{y}{13}$

(3) Take the derivative for the above both sides:

$$\underset{12/13}{\cos\theta} \cdot \frac{d\theta}{dt} = \frac{1}{13} \cdot \underset{-12}{\frac{dy}{dt}} \Rightarrow \frac{d\theta}{dt} = \frac{1}{13}(-12) \cdot \frac{13}{12} = \boxed{-1 \text{ rad/sec}}.$$

Ex2) Find two positive numbers whose sum is 20 and whose product is as large as possible.

Solution: When you see this word, you need to find the maximization (max. points).

1. $P = xy$

2. $x + y = 20 \Rightarrow \boxed{y = 20 - x}$ on $[0, 20]$.

3. $P = x(20 - x) = \boxed{20x - x^2}$

Take the derivative of the above, we obtain:

$$P' = 20 - 2x = 0$$
$$2(10 - x) = 0$$
$$\boxed{x = 10} \quad \text{So, } y = 20 - 10 = \boxed{10}.$$

Closed Interval Method:

1. end points $\boxed{x = 0}$ and $\boxed{x = 20}$
2. $P' = 0 \Rightarrow \boxed{x = 10}$
3. P' always exists $\longrightarrow$ Plug in $\boxed{P = 20x - x^2}$

$0 \longrightarrow P(0) = 20(0) - (0)^2 = 0$

$10 \longrightarrow P(10) = 20(10) - (10)^2 = \boxed{100}$ absolute max.

$20 \longrightarrow P(20) = 20(20) - (20)^2 = 0$

First derivative Test

P' + + + + | − − − −

inc. (10) dec. → abs. max.

2nd derivative Test

$P'' = -2 < 0$ So, it's max.

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