Study Guide 1

MATH 140 Lab: Section 1

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Note: This study guide contains my practice questions that I think will be useful for preparing you for the first exam in Calculus for Life Scientists.

Question 1: Find the following limits. Show your work.

a. $\lim_{x\to 2}\left(\frac{x^2-5x+6}{x^2-2x}\right)$ $\frac{0}{0}$

$$\Rightarrow \lim_{x\to 2}\frac{(x-3)\cancel{(x-2)}}{x\cancel{(x-2)}} = \lim_{x\to 2}\frac{x-3}{x} = \frac{2-3}{2} = \boxed{\frac{-1}{2}}$$

b. $\lim_{x\to 1}\left(\frac{\sqrt{x}-x}{1-\sqrt{x}}\right)$ $\frac{0}{0}$ We use conjugate method

$$\lim_{x\to 1}\left[\left(\frac{\sqrt{x}-x}{1-\sqrt{x}}\right)\cdot\frac{(\sqrt{x}+x)(1+\sqrt{x})}{(\sqrt{x}+x)(1+\sqrt{x})}\right] = \lim_{x\to 1}\left(\frac{(x-x^2)(1+\sqrt{x})}{(1-x)(\sqrt{x}+x)}\right) =$$

$$= \lim_{x\to 1}\left(\frac{x\cancel{(1-x)}(1+\sqrt{x})}{\cancel{(1-x)}(\sqrt{x}+x)}\right) = \lim_{x\to 1}\left(\frac{x(1+\sqrt{x})}{\sqrt{x}+x}\right) = \frac{1(1+\sqrt{1})}{\sqrt{1}+1} = \frac{2}{2} = \boxed{1}$$

c. $\lim_{x\to 1}\frac{x-1}{|x-x^2|}$ $\frac{0}{0}$ We find the limit from both sides

$$\lim_{x\to 1^+}\frac{(x-1)}{-(x-x^2)} = \lim_{x\to 1^+}\frac{(x-1)}{(x^2-x)} = \lim_{x\to 1^+}\frac{\cancel{(x-1)}}{x\cancel{(x-1)}} = \lim_{x\to 1^+}\frac{1}{x} = \frac{1}{1} = \boxed{1}$$

$$\lim_{x\to 1^-}\frac{(x-1)}{(x-x^2)} = \lim_{x\to 1^-}\frac{\overset{-1}{\cancel{(x-1)}}}{x\cancel{(1-x)}} = \lim_{x\to 1^-}\frac{-1}{x} = \frac{-1}{1} = \boxed{-1}$$

Since $\lim_{x\to 1^+} f(x) \neq \lim_{x\to 1^-} f(x)$, then $\lim_{x\to 1} f(x) = \text{DNE}$

where $f(x) = \frac{x-1}{|x-x^2|}$

d. $\lim\limits_{x\to 0}\frac{3-3\cos^2 x}{4x^2}$ $\quad\frac{0}{0}\quad$ $\sin^2 x+\cos^2 x=1 \Rightarrow \boxed{\sin^2 x = 1-\cos^2 x}$

$$\Rightarrow \lim_{x\to 0}\frac{3(1-\cos^2 x)}{4x^2} = \lim_{x\to 0}\frac{3(\sin^2 x)}{4x^2} = \lim_{x\to 0}\frac{3\,(\sin x)(\sin x)}{4\,(x)\cdot(x)} =$$

$$\underset{=1}{\frac{\sin x}{x}} \qquad \underset{=1}{\frac{\sin x}{x}}$$

$$= \lim_{x\to 0}\frac{3}{4} = \boxed{\frac{3}{4}}$$

e. $\lim\limits_{x\to -1^+}\frac{1-x}{(x+1)^2}$

$$\Rightarrow \lim_{x\to -1^+}\frac{1-x}{(x+1)^2} = \frac{1--1}{(-1+1)^2} = \frac{2}{0^+} = \boxed{+\infty}$$

f. $\lim\limits_{x\to 2^-}\frac{x}{x-2}$

$$\Rightarrow \lim_{x\to 2^-}\frac{x}{x-2} = \frac{2}{2-2} = \frac{2}{0^-} = \boxed{-\infty}$$

g. $\lim\limits_{x\to\infty}\frac{2x^2-1}{4x^3-5x-1}$ Using leading terms

$$\Rightarrow \lim_{x\to\infty}\frac{2\cancel{x^2}}{4x^{\cancel{3}\,1}} = \lim_{x\to\infty}\frac{2}{4x} = \frac{2}{4}\left(\frac{1}{\infty}\right) = \frac{2}{4}(0) = \boxed{0}$$

Check: The power of num. < the power of den.

$\Rightarrow$ So, the answer is zero.

h. $\lim\limits_{x\to 0^+} e^{\left(-\frac{1}{x}\right)}$

$$\Rightarrow \lim_{x\to 0^+} e^{\left(\frac{-1}{x}\right)} = e^{-\frac{1}{0^+}} = e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = \boxed{0}$$

i. $\lim_{x\to 0} \frac{\sin(x)+3x}{2x}$ $\frac{0}{0}$ Let's divide the num. & den. by x.

$$\Rightarrow \lim_{x\to 0} \frac{\underbrace{\frac{\sin(x)}{x}}_{1} + \frac{3\cancel{x}}{\cancel{x}}}{\frac{2\cancel{x}}{\cancel{x}}} = \lim_{x\to 0} \frac{1+3}{2} = \lim_{x\to 0} \frac{4}{2} = \frac{4}{2} = \boxed{2}$$

j. $\lim_{x\to -\infty} \frac{2x}{\sqrt{x^2+4}}$

$$\Rightarrow \lim_{x\to -\infty} \frac{2x}{\sqrt{x^2\left(1+\frac{4}{x^2}\right)}} = \lim_{x\to -\infty} \frac{2\cancel{x}}{\cancel{x}\sqrt{\left(1+\frac{4}{x^2}\right)}} = \lim_{x\to -\infty} \frac{2}{\sqrt{1+\frac{4}{x^2}}} =$$

$$= \lim_{x\to -\infty} \frac{2}{\sqrt{1+\underbrace{\frac{4}{\infty}}_{=0}}} = \lim_{x\to -\infty} \frac{2}{\sqrt{1+0}} = \frac{2}{\sqrt{1}} = \boxed{2}$$

Question 2: Find $f'(1)$ and $h'(2)$ using the definition of derivative where:

$$f(x) = \frac{x}{x+1} \quad \text{and} \quad h(x) = \sqrt{x-1}$$

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

$$\Rightarrow f'(1) = \lim_{x\to 1} \frac{\left(\frac{x}{x+1} - \frac{1}{2}\right)}{(x-1)} = \lim_{x\to 1} \frac{\frac{2x-(x+1)}{2(x+1)}}{(x-1)} = \lim_{x\to 1} \frac{\frac{\overbrace{2x-x}^{x}-1}{2(x+1)}}{(x-1)}$$

$$= \lim_{x\to 1} \frac{\frac{x-1}{2(x+1)}}{\frac{(x-1)}{1}} = \lim_{x\to 1} \left[\frac{\cancel{(x-1)}}{2(x+1)} \cdot \frac{1}{\cancel{(x-1)}}\right] = \lim_{x\to 1} \frac{1}{2(x+1)} = \frac{1}{2(1+1)} = \boxed{\frac{1}{4}}$$

$$h'(a) = \lim_{x\to a} \frac{h(x)-h(a)}{x-a}$$

use conjugate method

$$h'(2) = \lim_{x\to 2} \frac{\sqrt{x-1}-1}{x-2} = \lim_{x\to 2} \left[\left(\frac{\sqrt{x-1}-1}{x-2}\right) \cdot \left(\frac{\sqrt{x-1}+1}{\sqrt{x-1}+1}\right)\right] =$$

$$= \lim_{x\to 2} \left[\frac{x-1-1}{(x-2)(\sqrt{x-1}+1)}\right] = \lim_{x\to 2} \left(\frac{\cancel{(x-2)}}{\cancel{(x-2)}(\sqrt{x-1}+1)}\right) = \lim_{x\to 2} \frac{1}{(\sqrt{x-1}+1)}$$

$$= \frac{1}{\sqrt{2-1}+1} = \frac{1}{1+1} = \boxed{\frac{1}{2}}$$

Question 3: Discuss the continuity at $x = 2$ for the following function:

$$f(x) = \begin{cases} 3, & x = 2 \\ 3x - 2, & x > 2 \\ x^2, & x < 2 \end{cases}$$

① $f(2) = 3$

② $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (3x-2) = 3(2) - 2 = 6 - 2 = 4$

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x^2) = (2)^2 = 4$

$\Rightarrow \lim_{x \to 2} f(x) = 4$

$f(2) \neq \lim_{x \to 2} f(x)$

$3 \neq 4 \Rightarrow$ The function is not continuous at $\boxed{x = 2}$.

Question 4: Find the equation of the tangent line to the curve: $\boxed{y = 4\sqrt{x} - 2x}$ at $x = 4$.

$y - y_1 = m(x - x_1)$

$\boxed{y = 4x^{1/2} - 2x}$

x_1 y_1
$(4, 0)$

$m = y' \Rightarrow y' = 4\left(\frac{1}{2}\right)x^{-1/2} - 2$

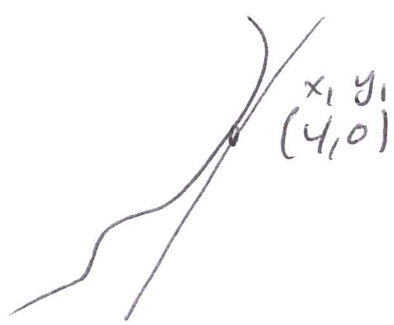
$\Rightarrow \boxed{y' = 2x^{-1/2} - 2}$

$y|_{x=4} = 4(4)^{1/2} - 2(4) = 8 - 8 = 0$

$y'|_{x=4} = 2(4)^{-1/2} - 2 = \frac{2}{\sqrt{4}} - 2 = \frac{\cancel{2}}{\cancel{2}} - 2 = \boxed{-1}$

$y - 0 = -1(x - 4)$

$\boxed{y = -x + 4}$



Question 5: Find y' without simplifying your answer for the following:

a. $y = 12x - x^2 - \frac{3}{\sqrt{x}}$

$\Rightarrow y = 12x - x^2 - 3x^{-1/2}$

$\Rightarrow y' = 12 - 2x - 3\left(\frac{-1}{2}\right)x^{-3/2}$

$\Rightarrow \boxed{y' = 12 - 2x + \frac{3}{2}x^{-3/2}}$

b. $y = x(3x^2 - \sqrt{x})$ $\Rightarrow y = x(3x^2 - x^{1/2}) = 3x^3 - x^{3/2}$

$\Rightarrow \boxed{y' = 9x^2 - \frac{3}{2}x^{1/2}}$

c. $y = \frac{2}{x^4} - x^3 + 2$

$\Rightarrow y = 2x^{-4} - x^3 + 2$

$\Rightarrow \boxed{y' = -8x^{-5} - 3x^2}$

d. $y = \frac{x^3}{(x^2+4)^2}$ Quotient Rule

$$y' = \frac{(x^2+4)^2(3x^2) - x^3[2(x^2+4)(2x)]}{[(x^2+4)^2]^2}$$

$$\boxed{y' = \frac{3x^2(x^2+4)^2 - 4x^4(x^2+4)}{(x^2+4)^4}}$$

e. $y = e^{\sin(2x)}$

$$y' = \left(e^{\sin(2x)}\right) \cdot \left(\cos(2x)\right)(2) = 2\cos(2x)\, e^{\sin(2x)}$$

f. $y = \ln(\sin(x^2))$

$$y' = \frac{1}{\sin(x^2)} \cdot \left(\cos(x^2)\right) \cdot (2x) = \frac{2x\cos(x^2)}{\sin(x^2)}$$

g. $y = (x^2)^x \Rightarrow y = x^{2x}$ Now, we take ln from both sides

$$\ln y = \ln x^{2x} \Longrightarrow \ln y = 2x \ln x \Longrightarrow \frac{1}{y} \cdot y' = (2)(\ln x) + \frac{2\cancel{x}}{\cancel{x}}$$

$$\Rightarrow y \cdot \frac{y'}{y} = (2\ln x + 2) \cdot y \Longrightarrow y' = y(2\ln x + 2) = \boxed{x^{2x}(2\ln x + 2)}$$

$$\Longrightarrow \boxed{y' = 2x^{2x}(\ln x + 1)}$$

Question 6: Find the equation of the tangent line to the curve: $y = \sin(4x)$ at $x = \frac{\pi}{8}$.

$$y = \sin 4x \Longrightarrow y\left(\tfrac{\pi}{8}\right) = \sin 4\left(\tfrac{\pi}{8}\right) = \sin\left(\tfrac{\pi}{2}\right) = 1$$

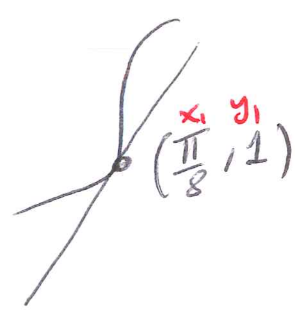
$\left(\overset{x_1}{\tfrac{\pi}{8}}, \overset{y_1}{1}\right)$

$$y' = m \Longrightarrow y' = \cos(4x) \cdot 4 \Longrightarrow y'\left(\tfrac{\pi}{8}\right) = \cos\left(4 \cdot \tfrac{\pi}{8}\right) \cdot 4 = 4\cos\left(\tfrac{\pi}{2}\right) \quad 0$$

$$\boxed{y'\left(\tfrac{\pi}{8}\right) = 0}$$

$$y - y_1 = m(x - x_1)$$

$$y - 1 = 0\left(x - \tfrac{\pi}{8}\right) \Rightarrow y - 1 = 0 \Longrightarrow \boxed{y = 1}$$



Question 7: Consider the function: $g(x) = \begin{cases} 1 + mx^2, & x < 1 \\ x^2 + mx, & x \geq 1 \end{cases}$

a. Show that $g(x)$ is continuous at $x = 1$.

To show $g(x)$ is continuous at $x=1$, it must have: $\lim_{x\to 1^+} g(x) = \lim_{x\to 1^-} g(x) = g(1)$

$\lim_{x\to 1} g(x)$: $\lim_{x\to 1^+}(x^2+mx) = 1+m$, $\lim_{x\to 1^-}(1+mx^2) = 1+m$ $\Rightarrow \lim_{x\to 1} g(x) = \boxed{1+m}$

$f(1) = (1)^2 + m(1) = \boxed{1+m} \Rightarrow \boxed{1+m = 1+m}$ ✓ $\boxed{\text{So, } g(x) \text{ is continuous at } x=1.}$

b. Find the value of m for which ~~f~~ $g(x)$ is differentiable at $x = 1$.

Since $g(x)$ is differentiable at $x=1$, then we obtain:

$g'(x) = \begin{cases} 2mx, & x<1 \\ 2x+m, & x \geq 1 \end{cases}$

$g'_+(1) = 2(1)+m = \boxed{2+m}$

$g'_-(1) = 2m(1) = \boxed{2m}$

$g'_+(1) = g'_-(1)$

$2+m = 2m$

$2+m-2m = 0$

$2-m=0 \Rightarrow \boxed{m=2}$

Question 8: Find all points of discontinuity for the following functions

a. $h(x) = \begin{cases} \frac{x^2-3x}{x-3}, & x \geq 3 \\ x+1, & x < 3 \end{cases}$

① $(x+1)$ is a continuous function. Hence, there is no discontinuous points. So, Domain: $(-\infty, \infty)$ "Everywhere"

② $\frac{x^2-3x}{x-3} \Rightarrow x-3=0 \Rightarrow \boxed{x=3}$

is continuous everywhere except $x=3$. So, Domain: $(-\infty,\infty)/x=3$.

b. $m(x) = \frac{3}{|2x|+4}$

$|2x|+4 \neq 0$

m(x) is continuous everywhere

So, Domain: $(-\infty, \infty)$.

Question 9: Consider the function: $f(x) = \begin{cases} \frac{1}{x^3+1}, & x < -1 \\ 2x+1, & -1 \leq x < 1 \\ 3x^2, & 1 < x \leq 2 \\ x^3, & x > 2 \end{cases}$

Find the following limits or state that the limit does exist. Explain why.

a. $\lim_{x\to-\infty} f(x) = \frac{1}{(-\infty)^3+1} = \frac{1}{-\infty} = \boxed{0}$

b. $\boxed{\lim_{x\to1} f(x) = 3} \Rightarrow \lim_{x\to1^-} f(x) = \lim_{x\to1^-}(2x+1) = 2(1)+1 = 3$

$\lim_{x\to1^+} f(x) = \lim_{x\to1^+}(3x^2) = 3(1)^2 = 3$

$3 = 3$ $\Bigg\} \lim_{x\to1} f(x) = \boxed{3}$

c. $\boxed{\lim_{x\to-1} f(x) = DNE} \Rightarrow \lim_{x\to-1^-} f(x) = \lim_{x\to-1} \frac{1}{x^3+1} = \frac{1}{-1+1} = \frac{1}{0} = \pm\infty$

$\neq$

$\lim_{x\to-1^+} f(x) = \lim_{x\to-1^+} 2x+1 = 2(-1)+1 = -2+1 = -1$

So, $\lim_{x\to-1} f(x) = DNE$

d. $\boxed{\lim_{x\to2} f(x) = DNE}$

$\lim_{x\to2^-} f(x) = \lim_{x\to2^-} 3x^2 = 3(2)^2 = 3(4) = 12$

$\neq$

$\lim_{x\to2^+} f(x) = \lim_{x\to2^+} x^3 = (2)^3 = 8$

So, $\lim_{x\to2} f(x) = DNE$

Question 10: Determine whether the following statements are true or false:

a. Assume that $f(x) = x^2 - 2\sqrt{x} - 1$is defined on $[1,2]$. Then, there exists a number c between 1 and 2 such that $f(c) = 0$. [......True............]

b. $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$ [...False..........]

c. $y = 1$ is a horizontal asymptote of $y = \frac{1}{x-1}$. [....False........]

$y = 0$ as $x \to \infty$

d. If $y = x^x$, $y' = x^x(\ln x + 1)$. [.......True.........]

$\ln y = x \ln x \Rightarrow \frac{1}{y} \cdot y' = (\ln x + 1) \Rightarrow y' = x^x(\ln x + 1)$

e. $x = -1$ is a vertical asymptote of $y = \frac{x-2}{x^2-x-2}$. [.......True.........]

Question 11: If $f(1) = 3$, $f'(1) = 2$, $g(1) = 10$, $g'(1) = 4$, $g'(3) = -2$, then find the value of $w'(1)$ where:

a. $w(x) = g(f(x))$

b. $w(x) = x^3 f(x)$

c. $w(x) = \ln(g(x)^2 + 1)$

a. $w'(x) = g'(f(x)) \cdot f'(x) \Rightarrow w'(1) = g'(f(1)) \cdot f'(1) = g'(3) \cdot (2)$

$w'(1) = (-2)(2) = \boxed{-4}$

b. $w'(x) = 3x^2 f(x) + x^3 f'(x) \Rightarrow w'(1) = 3f(1) + f'(1)$

$w'(1) = 3(3) + 2 = 9 + 2 = \boxed{11}$

c. $w'(x) = \frac{1}{g(x)^2 + 1} \cdot (2g'(x)g(x))$

$\Rightarrow w'(1) = \frac{1}{g(1)^2 + 1} \cdot (2g'(1)g(1))$

$\Rightarrow w'(1) = \frac{1}{100 + 1} \cdot (2(4)(10)) = \boxed{\frac{80}{101}}$

Question 12: Find the equilibrium at $c_1 = 1$ for the following difference equation:

$$c_{n+1} = \sqrt{c_n + 2}$$

$\Rightarrow c = \sqrt{c+2} \Rightarrow c^2 = c+2 \Rightarrow c^2 - c - 2 = 0$

$(c+1)(c-2) = 0 \Rightarrow \boxed{c=-1}$ or $\boxed{c=2}$

$c_1 = \sqrt{-1+2} = \sqrt{1} = 1$

$c_2 = \sqrt{c_1+2} = \sqrt{1+2} = \sqrt{3}$

$c_3 = \sqrt{c_2+2} = \sqrt{\sqrt{3}+2}$ Keeps increasing

So, $\lim_{n\to\infty} c_n = 2$

monotone convergent.

Question 13*: Assume that the height of a falling object t seconds after being dropped from a height of 64 feet can be written as follows:

$$h(t) = 64 - 16t^2 \text{ feet}$$

a. Find the average velocity between times $t = 1$ and $t = 2$.
b. Find the instantaneous velocity at times $t = 2$.

a. Average velocity $= \dfrac{h(2)-h(1)}{2-1} = \dfrac{(64-16(4))-(64-16)}{1}$

$= \boxed{-48 \text{ ft/s}}$

b. Instantaneous velocity at $t=2 = \lim_{t\to 2} \dfrac{h(t)-h(2)}{(t-2)} =$

$= \lim_{t\to 2} \dfrac{64-16t^2-0}{t-2} = \lim_{t\to 2} \dfrac{16(4-t^2)}{(t-2)} = \lim_{t\to 2} \dfrac{16(\cancel{2-t})(2+t)}{(\cancel{t-2})}$

$= \lim_{t\to 2} -16(2+t)$

$= -16(4) = \boxed{-64 \text{ ft/s}}$

***Reference:** Calculus: Early Transcendental Functions by Smith Minton 3rd Edition

Good Luck in Exam 1

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