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Section: Solution

Practice with Integrals Notes

Review:

- **Useful Integrals and Identities**

$$\int \sec^2(x)dx = \tan(x) + C \quad \bullet \quad \int \sec(x)\tan(x)dx = \sec(x) + C$$

$$\int \cosh(x)dx = \sinh(x) + C \quad \bullet \quad \int \sinh(x)dx = \cosh(x) + C$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right) + C \quad \bullet \quad \int \frac{1}{x^2 + a^2}dx = \frac{1}{a}tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \cot(x)dx = \ln|\sin(x)| + C \quad \bullet \quad \int \tan(x)dx = \ln|\sec(x)| + C$$

$$\int a^x dx = \frac{a^x}{\ln(a)} + C \quad \bullet \quad \int \frac{1}{x}dx = \ln|x| + C$$

$$\int \tan(x)dx = \ln|\sec(x)| + C \quad \bullet \quad \int \sec(x)dx = \ln|\sec(x) + \tan(x)| + C$$

$$\sin^2(x) = \frac{1}{2}(1 - \cos(2x)) \quad \bullet \quad \cos^2(x) = \frac{1}{2}(1 + \cos(2x))$$

$$\sin^2(x) + \cos^2(x) = 1 \quad \bullet \quad 1 - \sin^2(x) = \cos^2(x)$$

$$1 + \tan^2(x) = \sec^2(x) \quad \bullet \quad \sec^2(x) - 1 = \tan^2(x)$$

- **The Substitution Rule (u substitution using $u = g(x)$).**

Indefinite

$$\int f(g(x))\, g'(x)dx = \int f(u)du$$

Definite

$$\int_a^b f(g(x))\, g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Parameters: f must be continuous on the range of $u = g(x)$ <u>and</u> either g'(x) is continuous on the interval [a,b] (Definite Case) or g(x) is differentiable (Indefinite Case).

- **Integration by Parts** "Reversing the product rule"

$f'(x) = \frac{d\cancel{x}}{dx} = \frac{f(x)}{dx}$

Let f and g be differentiable functions, then the product rule states

$\frac{d}{dx}[f(x)g(x)] =$ $f'(x)g(x) + f(x)g'(x)$

Integrating both sides and splitting the integral implies:

$f(x)g(x) =$ $\int f'(x)g(x)dx + \int f(x)g'(x)dx = \int \frac{f(x)}{dx} \cdot g(x)dx + \int f(x)\frac{g(x)}{dx}dx$
$= \int f'(x)g(x) + \int g'(x)f(x)$

If we let $u = f(x)$, and $v = g(x)$ then $du = f'(x)dx$, and $dv = g'(x)dx$. With a little rearranging we get

$$\boxed{\int udv = uv - \int vdu}$$

Integration Strategy:
If you are given an integral and not told what technique of integration to use, then how do you recognize what technique to use? The following is some helpful advice and practice to help you answer this question and avoid the dreaded panic attack on your next exam.

- At this level the purpose of the Substitution Rule is to convert an unfamiliar integral into one that is familiar. The trick to using the rule successfully is to look for a function, possibly composed within another function, **with** its derivative laying around. For instance, in the integral (22 in the book) $\int \frac{\sin(\ln(x))}{x}dx$, we see the natural log sitting inside the sine function, multiplied by its derivative $\frac{1}{x}$. So we might try using Substitution as follows:
Let $u = \ln(x)$ then $du = \frac{1}{x}dx$ and we have

$$\begin{aligned}\int \frac{\sin(\ln(x))}{x}dx &= \int \sin\left([\ln(x)]\right) * \left[\frac{1}{x}dx\right] \\ &= \int \sin([u])[du] \\ &= -\cos([u]) + C \\ &= -\cos([\ln(x)]) + C\end{aligned}$$

- The trick to using Int. by Parts efficiently is a 'wise' choice of "u" and "dv". That wise choice comes down to intuitively choosing your "u" as the function that will simplify and eventually go away after repeated differentiation (like x^k for some natural number k), and a "dv" that won't get too messy when you integrate it (you have to know how to integrate it of course). Sometimes we use integration by parts on single functions that we don't know the integral of, but we know their derivatives as the following example shows.
Let $u = \sin^{-1}(x)$ and $dv = 1dx$, then $du = \frac{1}{\sqrt{1-x^2}}$ and $v = x$. Thus,

$$\int_0^1 \sin^{-1}(x)dx = \left[\sin^{-1}(x) * x\right]_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}}dx$$

Using U-sub with $u = 1 - x^2$ and $du = -2x\ dx$ we get,

$$\begin{aligned}\left[\frac{\pi}{2} - 0\right] - \int_{1-(0)^2}^{1-(1)^2} \frac{\frac{-1}{2}(-2x)}{\sqrt{1-x^2}}dx &= \frac{\pi}{2} - \frac{1}{2}\int_0^1 \frac{du}{\sqrt{u}} \\ \Rightarrow \frac{\pi}{2} - \frac{1}{2}\left[2\sqrt{u}\right]_0^1 &= \frac{\pi}{2} - \frac{1}{2}[2-0] \\ &= \frac{\pi}{2} - 1\end{aligned}$$

- The main idea to remember when encountering integrals of the form $\int[\sin(x)]^n[\cos(x)]^m dx$ or $\int[\tan(x)]^n[\sec(x)]^m dx$, n, m whole numbers, is to treat them like you treat integrals involving U-Substitution, because you will eventually be using U-Substitution to solve them. The difference is that you might have to use the half angle formulas or the Fundamental Trig Identity beforehand. Knowing when is a matter of practice and familiarity.

- The main idea for the other types of Trig-Sub is knowing when to substitute the variable for a trig function. This will happen almost always when there is an integral with the expression $\sqrt{ax^2+b}$ in there somewhere for real numbers a and b. The easiest way to do these problems is to memorize the three trig id's above that you get from the Fundamental Trig Identity and making the expression under the square root look like one of them.

- The main idea concerning partial fractions is to transform "hard" rational integrals into the sum of integrals of "easier" rational function using the technique of Partial Fractions. Here is an example of using partial fractions to integrate:

$$\int \frac{2}{(x+1)^2(x-1)}dx$$

Since we do not know how to inegrate this, we break down our integrand into easier fractions:

$$\frac{2}{(x+1)^2(x-1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-1}$$

Now we need to solve for A, B, and C as following:

$$A(x+1)(x-1) + B(x-1) + C(x+1)^2 = A(x^2-1) + B(x-1) + C(x^2+2x+1)$$
$$= x^2(A+C) + x(B+2C) + (-A-B+C) = 2$$

Therefore, we have

$$A + C = 0, B + 2C = 0, -A - B + C = 2$$

Solving this set of equations, one can find $A = -\frac{1}{2}, B = -1, C = \frac{1}{2}$. Thus, the given integral is equal to

$$\int \frac{-1}{2(x+1)}dx + \int \frac{-1}{(x+1)^2}dx + \int \frac{1}{2(x-1)})dx$$

- There are two main types of improper integrals: **Infinite Interval** and **Infinite Discontinuities** (you could of course have both in the same problem). The first is easy to spot because one of the limits of the integral is $\pm\infty$. The other is a bit harder to spot at first, but after some practice you will learn to anticipate them. There are two main forms that integrals involving Infinite Discontinuities arise in this class. The first is when you are asked to integrate over an interval that contains a number that makes the denominator of the integrand 0. The second is when you are asked to integrate a function (usually functions like $\tan(\theta)$ or $\frac{1}{x}$ but not limited to) that has a infinite discontinuity "built in."

Practice

1. Simplify the integrand if possible first.

$$\int \sqrt{x}(1+\sqrt{x}))dx$$

$\Rightarrow \int \sqrt{x}(1+\sqrt{x})dx = \int(\sqrt{x}+x)dx = \int(x^{\frac{1}{2}}+x)dx = \int x^{1/2}dx + \int x\,dx =$

$= \frac{x^{3/2}}{3/2} + \frac{x^2}{2} + C$

$= \boxed{\frac{2}{3}x^{3/2} + \frac{x^2}{2} + C}$

2. Look for an obvious substitution instead of trying trigonometric substitution.

$$\int \frac{x}{x^2+1}dx$$

$u = x^2+1$

$du = 2x\,dx \Rightarrow \boxed{x\,dx = \frac{du}{2}}$

$\frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln|u| + C$

$= \frac{1}{2}\ln|x^2+1| + C$

$\frac{1}{2}\ln|x^2+1| + C$

3. Classify, but do not integrate, the integrals below according to its form as follows:

A. If the integrand is a product of $\sin(x)$ and $\cos(x)$, etc, use techniques learned in 'trigonometric integrals'

B. If the integrand is a rational function, consider partial fractions

C. If the integrand looks like (a) a product or (b) a function that simplifies on differentiation (like log or inverse trig) then consider parts.

D. If the integrand involves roots of expressions like $(a^2 - x^2)$, etc., consider trig. substitution (but only after looking for an obvious substitution).

E. If the integrand can be simplified into an easily integrable form using substitution.

(a) Assuming $a \in \mathbb{R}$

$$\int_{-\frac{a}{2}}^{\frac{a}{2}} \frac{1}{(x+a)^{\frac{1}{3}}} dx$$ Class E

Class: E

(b)

$$\int \frac{4x+1}{(x^2-9)} dx$$ Class B

Class: B

(c)

$$\int \frac{4}{\sqrt{3x^2+4}} dx$$ Class D

Class: D

(d)

$$\int_{-2}^{0} \frac{1}{(x-\frac{2}{3})^{\frac{1}{3}}} dx$$ Class E

Class: E

(e)

$$\int_{0}^{3} -x\cos(x^2) dx$$ Class C

Class: C

(f)

$$\int_{0}^{\pi} \sin^3(x)\cos^2(x) dx$$ Class A

Class: A

4. Reverse Engineer (create) an indefinite integral that **must** be solved using integration by parts at least twice. Write out the problem and the solution to the integral below.
Hint: Using one or two of the following functions in your integral may be helpful: $e^x, \sin(x), \cos(x), \sinh(x), \cosh(x)$, etc..

$\int \underbrace{x^2}_{u} \underbrace{\cos(x)\,dx}_{dv}$ ⇐ Problem

Solution: $u = x^2$ $\quad dv = \cos(x)\,dx$

$du = 2x\,dx$ $\quad v = \sin(x)$

$$\Rightarrow \int x^2 \cos(x)\,dx = x^2 \sin(x) - \int 2x \sin(x)\,dx$$

$$= x^2 \sin(x) - 2\int x \sin(x)\,dx$$

thus, $\int x^2 \cos(x)\,dx = \boxed{x^2 \sin(x) + 2x\cos(x) - 2\sin(x) + C}$

$\int x \sin(x)\,dx$

$u = x$ $\quad dv = \sin(x)\,dx$

$du = dx$ $\quad v = -\cos(x)$

$\int x \sin(x)\,dx = -x\cos(x) + \int \cos(x)\,dx$

$= \boxed{-x\cos(x) + \sin(x) + C}$