Mohammed K A Kaabar — Differentiation Theorems — March 2nd, 2016

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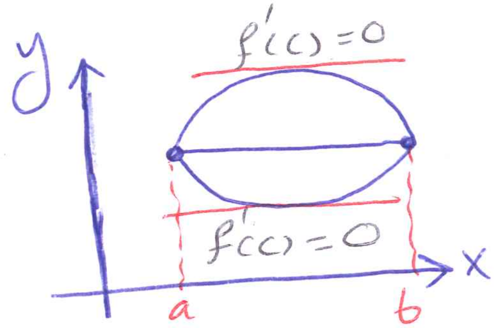
* Rolle's Theorem:

$f(x)$ is defined on $[a,b]$ and satisfies the following conditions:



1. f is continuous in $[a,b]$
2. f is differentiable in (a,b)
3. $f(a) = f(b)$

Then, there exists at least one c in (a,b) such that $f'(c) = 0$

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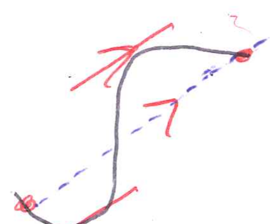
* Mean Value Theorem:

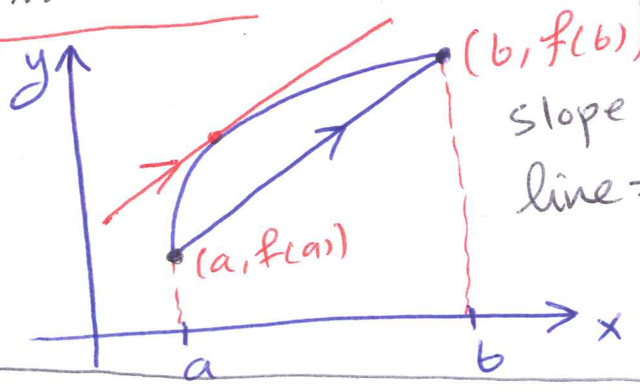
$f(x)$ is defined on $[a,b]$ and satisfies the following conditions:

1. f is continuous in $[a,b]$
2. f is differentiable in (a,b).



Then, there exists a c in (a,b) such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



slope of secant line $= \frac{f(b) - f(a)}{b - a}$

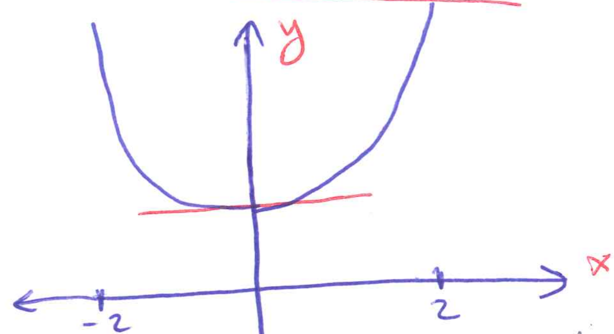
Ex1 Given: $f(x)=x^2+1$ defined on $[-2,2]$. Find c using **Rolle's Theorem**.

Solution:

[1] f is continuous on $[-2,2]$ because $f(x)=x^2+1$ is a polynomial function.

[2] f is differentiable on $(-2,2)$

[3] $f(-2)=f(2)=5$



Then, there exists at least $c \in (-2,2)$ such that

$$f'(c)=0 \Rightarrow 2c=0 \Rightarrow \boxed{c=0}$$

Ex2 Given: $f(x)=x^3+x^2$ defined on $[-1,1]$. Find c using the **Mean Value Theorem**.

Solution:

[1] f is continuous on $[-1,1]$ because $f(x)=x^3+x^2$ is a polynomial function.

[2] $f'=3x^2+2x$ on $(-1,1)$.

Then, there exists a $c \in (-1,1)$ such that:

$$f'(c)=\frac{f(b)-f(a)}{b-a} \Rightarrow f'(c)=\boxed{3c^2+2c}$$

$$\Rightarrow \frac{f(b)-f(a)}{b-a}=\frac{f(1)-f(-1)}{1--1}=\frac{2-0}{2}=\boxed{1} \Rightarrow$$

$\Rightarrow 3c^2+2c=1 \Rightarrow 3c^2+2c-1=0$

$\Rightarrow (3c-1)(c+1)=0$

$\Rightarrow \boxed{c=\frac{1}{3}}$ or $\boxed{c=-1}$ ✗ ← because the mean value theorem says the inside of the given interval, but not the edge. So, we have to remove it.

So, $\boxed{c=\frac{1}{3}}$.

Ex3) Given: $f(x)=x^3-x^2-x+1$ defined on $[0,2]$. Find c using the Mean Value Theorem.

Solution:

1. f is continuous on $[0,2]$ (polynomial).
2. $f'=3x^2-2x-1$. $\Rightarrow$ There exists $c\in(0,2)$ such that $f'(c)=\frac{f(2)-f(0)}{2-0} \Rightarrow 3c^2-2c-1=\frac{3-1}{2-0}$

$\Rightarrow 3c^2-2c-1=1 \Rightarrow \boxed{3c^2-2c-2=0}$

Use: $\boxed{x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}}$

$c=\frac{2\pm\sqrt{4+24}}{6}$

$\Rightarrow \boxed{c=\frac{2\pm\sqrt{28}}{6}}$.

* **Theorem**: If $f'(x) = 0$ for all x in some open interval I, then $f(x) = C$ for all x in I.

Proof of this theorem:

let's assume: a and $b \in I$. So, f is defined on $[a,b]$.

Now, by using MVT, we obtain:

① f is continuous on $[a,b]$

② f is differentiable on (a,b)

③ $f'(c) = \frac{f(b)-f(a)}{b-a}$, $f'(c) = 0$ $\Rightarrow \frac{0}{1} \times \frac{f(b)-f(a)}{b-a}$

$\Rightarrow f(b) - f(a) = 0$

$\Rightarrow \boxed{f(a) = f(b)}$

* **Corollary**: If $f'(x) = h'(x)$ for all x in I, then $f(x) = h(x) + C$.

Proof:

let: $m(x) = f(x) - h(x) \Rightarrow m'(x) = f'(x) - h'(x)$

$\Rightarrow m'(x) = 0 \Rightarrow \boxed{m(x) = C}$

$f(x) - h(x) = C \Rightarrow \boxed{f(x) = h(x) + C}$

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