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* Definition: The derivative of $y=f(x)$, denoted by y' or $f'(x)$ or $\frac{dy}{dx}$, is defined as follows:

$$\boxed{f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}}$$ provided limit exists.

Ex1) Given $f(x)=x^2$. Find $f'(x)=?$

Solution:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$f(x+h)=(x+h)^2=x^2+2xh+h^2$$

$$f'(x)=\lim_{h\to 0}\frac{\cancel{x^2}+2xh+h^2-\cancel{x^2}}{h}=\lim_{h\to 0}\frac{\cancel{h}(2x+h)}{\cancel{h}}=$$

$$=\lim_{h\to 0}(2x+h)=2(x)+0=\boxed{2x}$$

So, $\boxed{f'(x)=2x}$.

Ex2) Given $f(x)=x^3$. Find $f'(1)=?$

			1		$n=0$
		1		1	$n=1$
	1	2		1	$n=2$
	1	3	3	1	$n=3$
1	4	6	4	1	$n=4$

Solution:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$f(x+h)=(x+h)^3$ To find $(x+h)^3$ we use Pascal's Triangle

So, $(x+h)^3=x^3+3x^2h+3xh^2+h^3$

$$\Rightarrow f'(x)=\lim_{h\to 0}\frac{\cancel{x^3}+3x^2h+3xh^2+h^3-\cancel{x^3}}{h}=\lim_{h\to 0}\frac{\cancel{h}(3x^2+3xh+h^2)}{\cancel{h}}$$

$$=3x^2+3x(0)+(0)^2=\boxed{3x^2}\Rightarrow f'(1)=3(1)^2=\boxed{3}.$$

Ex3 Given $f(x) = |x-2|$. Find $f'(2) = ?$

Solution:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \Rightarrow f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x-2} =$$

$$= \lim_{x \to 2} \frac{|x-2| - |2-2|}{x-2} = \lim_{x \to 2} \frac{|x-2|}{x-2}$$

* Do we need to discuss the limit from both sides?

Answer: Yes, because we can discuss the limit from both sides if we have one of the following:

1- Absolute value function
2- Roots function
3- Piecewise-defined function
4- $\frac{1}{0} = \pm\infty$

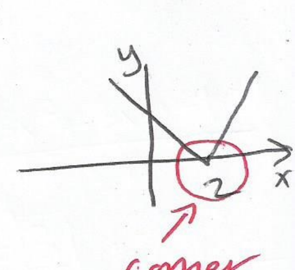
$$|x-2| = \begin{cases} (x-2), & x \geq 2 \\ -(x-2), & x < 2 \end{cases}$$

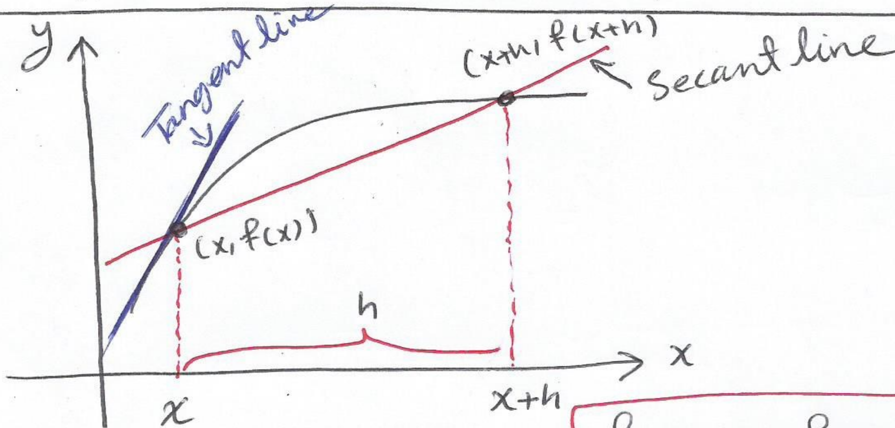
So, $\lim_{x \to 2} \frac{|x-2|}{x-2}$

$$\left.\begin{aligned} &\Rightarrow \lim_{x \to 2^+} \frac{\cancel{x-2}}{\cancel{x-2}} = \boxed{1} \\ &\Rightarrow \lim_{x \to 2^-} \frac{-\cancel{(x-2)}}{\cancel{(x-2)}} = \boxed{-1} \end{aligned}\right\} \Rightarrow \lim_{x \to 2^+} \frac{(x-2)}{(x-2)} \neq \lim_{x \to 2^-} \frac{-(x-2)}{(x-2)} \Rightarrow 1 \neq -1$$

This implies that $\lim_{x \to 2} \frac{|x-2|}{x-2}$ is DNE

So, there is no derivative at $x = 2$.



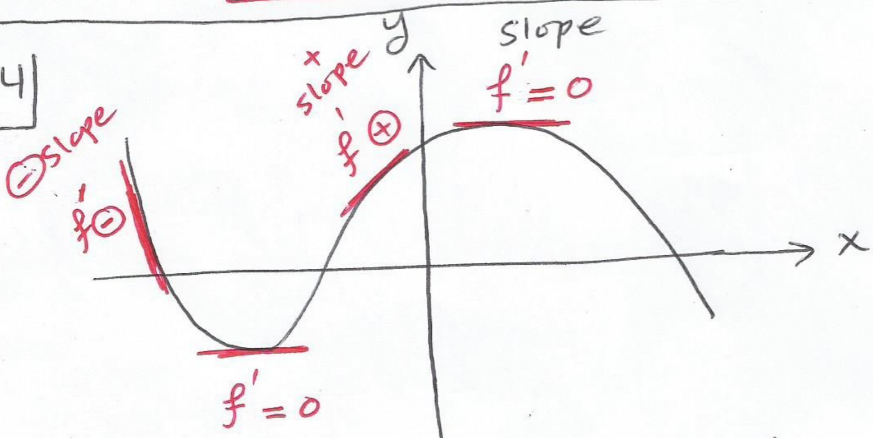


Slope of Secant line $= \boxed{\dfrac{f(x+h)-f(x)}{h}}$

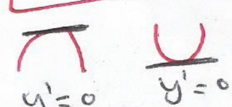
$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}}$$

Therefore, $\boxed{f'(x) \text{ is the slope of the tangent line at } x.}$

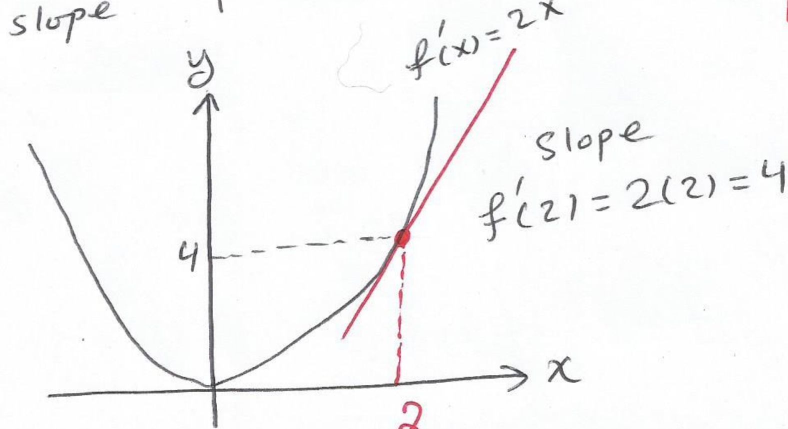
Ex4



Note: The horizontal tangent line is $\boxed{y' = f'(x) = 0}$



Ex5



Ex 6) Given: $f(x)=\sqrt{x^2+5}$.

Part a: $f'(2)=?$

Part b: Find the equation of the tangent line to the curve of $f(x)$ at $x=2$.

Solution:

Part a: $f'(a)=\lim\limits_{x\to a}\frac{f(x)-f(a)}{x-a}$

$$f'(2)=\lim_{x\to 2}\frac{f(x)-\boxed{f(2)}}{x-2} \longrightarrow 3$$

Conjugate

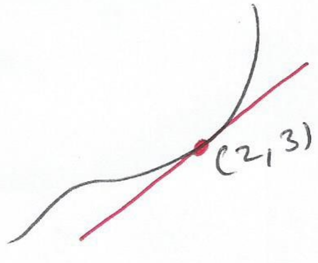
$$=\lim_{x\to 2}\left[\frac{\sqrt{x^2+5}-3}{x-2}\cdot\frac{\sqrt{x^2+5}+3}{\sqrt{x^2+5}+3}\right]=$$

$$=\lim_{x\to 2}\frac{x^2+5-9}{(x-2)(\sqrt{x^2+5}+3)}=\lim_{x\to 2}\frac{x^2-4}{(x-2)(\sqrt{x^2+5}+3)}=$$

$$=\lim_{x\to 2}\frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}(\sqrt{x^2+5}+3)}=\frac{(2+2)}{\sqrt{4+5}+3}=\frac{4}{\sqrt{9}+3}=\frac{4}{3+3}=\frac{4}{6}$$

$$=\boxed{\frac{2}{3}}$$

Part b:



$$y-y_1=m(x-x_1)$$

$$y-3=\frac{2}{3}(x-2)$$

$$y=\frac{2}{3}(x-2)+3$$

$$y=\frac{2}{3}x-\frac{4}{3}+3$$

$$\boxed{y=\frac{2}{3}x+\frac{5}{3}}$$