Mohammed K A Kaabar — Power Series — Date: Nov. 13th, 2015

Math 182

Honors Calculus II

---

* Power Series

Definition: A power series about $\boxed{x=0}$ has the form:

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots.$$

Definition: A power series about $\boxed{x=a}$ has the form:

$$\sum_{n=0}^{\infty} a_n (x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots$$

Example ①: Given: $1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 - \frac{1}{8}(x-2)^3 + \cdots$

Find the radius of convergence and interval of convergence.

Solution: power series about $a=2$

$r = \frac{-1}{2}(x-2)$

$a = 1$

To find the radius of convergence $\Rightarrow |r| < 1$

$\Rightarrow \left|\frac{-1}{2}(x-2)\right| < 1$

$\Rightarrow \frac{1}{2}|x-2| < 1$

$\Rightarrow |x-2| < 2$ Thus, the radius of convergence $= \boxed{R=2}$

$\Rightarrow -2 < x-2 < 2$

$\Rightarrow \boxed{0 < x < 4} \Leftarrow$ This is the interval of convergence.

---

* Three cases for radius of convergence and interval of convergence:

① Convergent on an interval:

$a < x < b$

$a \leq x < b$

$a < x \leq b$

$a \leq x \leq b$

So, $\boxed{R = a > 0}$

② Convergent always:

$-\infty < x < \infty$ (Everywhere)

$\boxed{R = \infty}$

③ Convergent only at $\boxed{x = 0}$:

$\boxed{R = 0}$

Example ②: Investigate: $\sum_{n=1}^{\infty} \frac{(-2)^n (x+3)^n}{\sqrt{n}}$. Then, find the radius of convergence and the interval of convergence.

Solution: $\sum_{n=1}^{\infty} \frac{(-2)^n (x+3)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n 2^n (x+3)^n}{\sqrt{n}}$

By ratio test: $\lim_{n \to \infty} \left| \frac{\overset{2}{\cancel{2^{n+1}}} \, \overset{(x+3)}{\cancel{(x+3)^{n+1}}}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{\cancel{2^n} \, \cancel{(x+3)^n}} \right|$

$= \lim_{n \to \infty} 2|x+3| \frac{\sqrt{n}}{\sqrt{n}+1} = 2|x+3| < 1$ Convergent $\Longrightarrow$ leading terms

$|x+3| < \frac{1}{2}$

So, $\boxed{R = \frac{1}{2}}$ = Radius of convergence

$-\frac{1}{2} < x+3 < \frac{1}{2}$

$-\frac{1}{2} - 3 < x < \frac{1}{2} - 3$

$\boxed{\frac{-7}{2} < x \leq \frac{-5}{2}}$

$x = \frac{-7}{2}$

$\sum_{n=1}^{\infty} \frac{(-1)^n 2^n \left(\frac{-1}{2}\right)^n}{\sqrt{n}} =$

$\sum_{n=1}^{\infty} \frac{(-1)^n \left(2 \cdot \frac{-1}{2}\right)^n}{\sqrt{n}} =$

$\sum_{n=1}^{\infty} \frac{(-1)^n (-1)^n}{\sqrt{n}} =$

$\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{\sqrt{n}} \approx \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} =$

$= \sum_{n=1}^{\infty} \frac{1}{n^{1/2}} \quad p = 1/2 < 1$

divergent by p-series.

$x = \frac{-5}{2}$

$\sum_{n=1}^{\infty} \frac{(-1)^n 2^n \left(\frac{1}{2}\right)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n \left(2 \cdot \frac{1}{2}\right)^n}{\sqrt{n}}$

$= \sum_{n=1}^{\infty} \frac{(-1)^n (1)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

(i) Alternating

(ii) $\lim_{n \to \infty} \frac{1}{\sqrt{n}} = \frac{1}{\infty} = 0$

(iii) $f(x) = x^{-1/2} \Rightarrow f'(x) = -\frac{1}{2} x^{-3/2} =$

$= \frac{-1}{2x^{3/2}} < 0$ decreasing

So, it's convergent by alternating series test

Thus, $\boxed{R = \frac{1}{2}}$ and Interval of convergence $= IC = \left(\frac{-7}{2}, \frac{-5}{2}\right]$.

* Power Series Representation:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

$$\boxed{\frac{a}{1-r}}$$

$$\boxed{\begin{array}{l} a = 1 \\ r = x \end{array}} \qquad -1 < x < 1$$

Example ③: Given: $f(x) = \frac{3}{2+x}$. Express it as a power series representation.

Solution:

$$\frac{3}{2+x} = 3 \cdot \frac{1}{2+x}$$

$$= \frac{3}{2} \cdot \frac{1}{1 + \left(\frac{x}{2}\right)} = \frac{3}{2} \cdot \frac{1}{1 - \left(-\frac{x}{2}\right)} =$$

$$\boxed{\begin{array}{l} a = 1 \\ r = -\frac{x}{2} \end{array}}$$

$$= \frac{3}{2}\left[1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \frac{x^4}{16} - \cdots\right]$$

$$= \frac{3}{2} \sum_{n=0}^{\infty} \left(\frac{-x}{2}\right)^n = \boxed{\frac{3}{2} \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^n}}$$

☒

Example ④: Given: $g(x) = \frac{1}{(1-x)^2}$. Express it as a power series representation.

Solution:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

Differentiate both sides, we obtain:

$$(1-x)^{-1\,\prime} = -(1-x)^{-2} \cdot (-1)$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots = \sum_{n=1}^{\infty} n x^{n-1}$$

$\Longrightarrow$

Example ⑤: Given: $h(x) = \ln(1-x)$. Express it as a power series representation.

Solution:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

Integrate both sides from 0 to x, we obtain:

$$\int_0^x \frac{1}{1-x}\,dx = \int_0^1 \left[1 + x + x^2 + x^3 + \cdots\right] dx$$

$$-\ln(1-x)\Big|_0^x = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$$

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$$

$$\Rightarrow \boxed{\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots}$$