



Final Exam Study Guide

MATH 172 Lab: Sections 7 and 8

Lab Instructor (TA): Mohammed Kaabar

Student's Name: Mohammed K A Kaabar

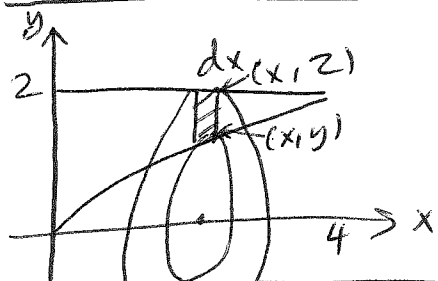
Student's ID: - Solution -

Note: This study guide contains comprehensive practice questions for the final exam in Calculus II.

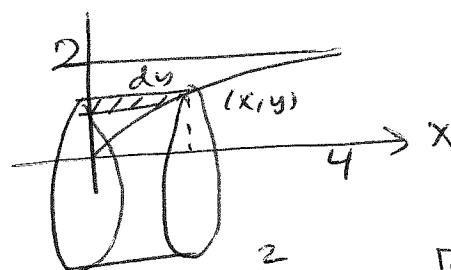
Question 1: Find the volumes of the region generated by revolving $y = \sqrt{x}$, $y = 2$, and y -axis about the following:

Part a: x -axis

Washer Method: $V = \int_0^4 (\pi(4)^2 - \pi(y)^2) dx$
 $= \pi \int_0^4 (4-x) dx = \boxed{8\pi}$



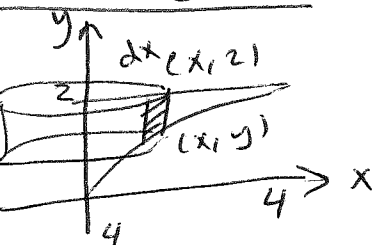
OR Shell Method



$V = \int_0^2 2\pi y x dy = 2\pi \int_0^2 y \cdot y^2 dy = \boxed{8\pi}$

Part b: y -axis

Shell Method:



$V = \int_0^4 2\pi x(2-y) dx = 2\pi \int_0^4 x(2-\sqrt{x}) dx = 2\pi \left(x^2 - \frac{x^{3/2}}{3/2} \right) \Big|_0^4 = 2\pi \left(\frac{16}{5} \right) = \boxed{\frac{32\pi}{5}}$

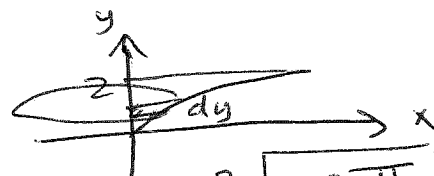
Part c: $y = -1$

Shell Method

$V = \int_0^2 2\pi(y+1)y^2 dy$
 $= 2\pi \int_0^2 (y^3 + y^2) dy = 2\pi \left(\frac{y^4}{4} + \frac{y^3}{3} \right) \Big|_0^2 = \boxed{\frac{40\pi}{3}}$

OR Disk Method:

$V = \int_0^2 \pi x^2 dy = \pi \int_0^2 y^4 dy = \pi \left(\frac{y^5}{5} \right) \Big|_0^2 = \boxed{\frac{32\pi}{5}}$



Question 2: Evaluate the following indefinite integrals:

Part a: $\int x^{-3} \ln(x) dx = \int \frac{\ln(x)}{x^3} dx$ by parts

$u = \ln x \rightarrow dv = x^{-3} dx$
 $du = \frac{1}{x} dx \leftarrow v = \frac{x^{-2}}{-2} = -\frac{1}{2x^2} \rightarrow x^{-3}$

$$\int \frac{\ln x}{x^3} dx = -\frac{1}{2x^2} \ln(x) + \frac{1}{2} \int \frac{1}{x^3} dx$$

$$= -\frac{1}{2x^2} \ln(x) + \frac{-1}{2x^2} \left(\frac{1}{2}\right) + C$$

$$= \boxed{-\frac{1}{2x^2} \ln(x) - \frac{1}{4x^2} + C} \quad \square$$

Part b: $\int \sin^5(x) \cos^3(x) dx$

$$\int \sin^5(x) \cos^2(x) \cos(x) dx = \int \sin^4(x) (1 - \sin^2(x)) \cos(x) dx$$

$$\boxed{u = \sin(x)} \\ \boxed{du = \cos(x) dx}$$

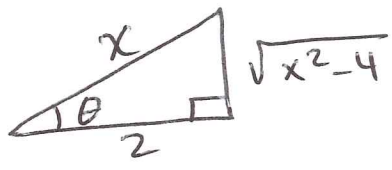
$$\int u^4 (1 - u^2) du = \int (u^4 - u^6) du = \frac{u^5}{5} - \frac{u^7}{7} + C =$$

$$\boxed{\sin^2 x + \cos^2 x = 1} \\ \boxed{\cos^2 x = 1 - \sin^2 x}$$

$$= \frac{\sin^5(x)}{5} - \frac{\sin^7(x)}{7} + C = \boxed{\frac{(\sin(x))^5}{5} - \frac{(\sin(x))^7}{7} + C} \quad \square$$

Part c: $\int \frac{dx}{\sqrt{x^2-4}}$

$x = 2 \sec \theta \Rightarrow \boxed{\sec \theta = \frac{x}{2}}$



$dx = 2 \sec \theta \tan \theta d\theta$

$\sqrt{x^2-4} = \sqrt{4 \sec^2 \theta - 4} = 2 \sqrt{\sec^2 \theta - 1} = 2 \sqrt{\tan^2 \theta} = 2 \tan \theta$

$$\boxed{1 + \tan^2 \theta = \sec^2 \theta} \\ \boxed{\sec^2 \theta - 1 = \tan^2 \theta}$$

$$\int \frac{2 \sec \theta \tan \theta d\theta}{2 \tan \theta} = \int \sec \theta d\theta$$

$$= \ln |\sec \theta + \tan \theta| + C$$

$$= \boxed{\ln \left| \frac{x}{2} + \frac{\sqrt{x^2-4}}{2} \right| + C} \quad \square$$

Part d: $\int \tan^{-1}(\sqrt{x}) dx$.

Let $u = \sqrt{x} \Rightarrow u^2 = x \Rightarrow \boxed{2u du = dx}$

$\int \tan^{-1}(u) 2u du = 2 \int u \tan^{-1}(u) du$ by parts

$\int x \tan^{-1} x dx$ $u = \tan^{-1} x \rightarrow dv = x dx$
 $du = \frac{1}{1+x^2} dx \leftarrow v = \frac{x^2}{2}$

$\int x \tan^{-1} x dx = \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} \int \frac{x^2(1-1)}{1+x^2} dx = \frac{1}{2} x^2 \tan^{-1}(x) - \frac{1}{2} \int \left[\frac{x^2+1}{x^2+1} - \frac{1}{1+x^2} \right] dx$

$= \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} \int dx + \frac{1}{2} \int \frac{1}{1+x^2} dx = \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} x + \frac{1}{2} \tan^{-1} x + C$

$\Rightarrow 2 \int u \tan^{-1} u du = u^2 \tan^{-1} u - u + \tan^{-1} u + C$

$\Rightarrow \boxed{\int \tan^{-1}(\sqrt{x}) dx = x \tan^{-1}(\sqrt{x}) - \sqrt{x} + \tan^{-1}(\sqrt{x}) + C}$

Part e: $\int \frac{x+2}{x^3+x} dx$.

$\frac{x+2}{x^3+x} = \frac{A}{x} + \frac{Bx+C}{x^2+1} \Rightarrow \frac{x+2}{x^3+x} = \frac{A(x^2+1) + (Bx+C)(x)}{x(x^2+1)}$

$x+2 = A(x^2+1) + (Bx+C)(x)$

$x=0: 2 = A \Rightarrow \boxed{A=2}$

$x=1: 3 = 2A + B + C$

$x=-1: 1 = 2A + (-B+C)(-1)$

$1 = 2A + (B-C)$

$1 = 2A + B - C$

$3 = 2A + B + C$

$1 = 2A + B - C$

$4 = 4A + 2B$

By substituting $A=2$ in $\boxed{4 = 4A + 2B}$

$4 = 8 + 2B \Rightarrow 4 - 8 = 2B \Rightarrow -4 = 2B$

$\Rightarrow \boxed{B = -2}$

$3 = 2(2) + (-2) + C \Rightarrow 3 = 4 - 2 + C$

$\Rightarrow 3 - 4 + 2 = C \Rightarrow \boxed{C = 1}$

So, $\int \frac{x+2}{x^3+x} dx = \int \left[\frac{2}{x} + \frac{-2x+1}{x^2+1} \right] dx =$

$\int \left[\frac{2}{x} - \frac{2x}{x^2+1} + \frac{1}{x^2+1} \right] dx = \boxed{2 \ln|x| - \ln|x^2+1| + \tan^{-1}(x) + C}$

Question 3: Evaluate the following definite integral:

$$\int_3^4 \frac{dx}{x^2 - 6x + 10}$$

by completing the square

$$x^2 - 6x + 10 = [x^2 - 6x + 9] + 10 - 9 = [(x-3)^2 + 1]$$

$$\begin{aligned} u &= x - 3 \\ du &= dx \end{aligned}$$

$$\int_3^4 \frac{dx}{[(x-3)^2 + 1]} = \int_0^1 \frac{du}{u^2 + 1} = \tan^{-1}(u) \Big|_0^1 =$$

$$= \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4} - 0 = \boxed{\frac{\pi}{4}}$$

□

$$\begin{aligned} \bullet \text{ when } x=4 &\rightarrow u=4-3=1 \\ &\Rightarrow \boxed{u=1} \\ \bullet \text{ when } x=3 &\rightarrow u=3-3=0 \\ &\Rightarrow \boxed{u=0} \end{aligned}$$

Question 4: Write out the partial fraction decomposition of the following expression. **DO NOT SOLVE FOR COEFFICIENTS:**

$$\frac{x^3 + 2x^2 - x + 1}{x(x^2 - 4)(x^2 + 6)^2}$$

$$\frac{x^3 + 2x^2 - x + 1}{x(x^2 - 4)(x^2 + 6)^2} = \frac{x^3 + 2x^2 - x + 1}{x(x-2)(x+2)(x^2 + 6)^2}$$

$$= \frac{A}{x} + \frac{B}{(x-2)} + \frac{C}{(x+2)} + \frac{Dx+E}{(x^2+6)} + \frac{Fx+G}{(x^2+6)^2}$$

□

Question 5: Evaluate the following improper integrals and determine whether the integral converges or diverges:

Part a:

$$\lim_{t \rightarrow \infty} \int_1^t x^2 e^{-x^3} dx$$

$$\lim_{t \rightarrow \infty} \left[\frac{-1}{3} e^{-t^3} + \frac{1}{3} e^{-1} \right]$$

$$= \frac{1}{3} e^{-1} = \frac{1}{3e} \approx 0.1226 \approx 0.12$$

Converges

□

$$\int_1^{\infty} \frac{e^{-x^3}}{x^2} dx = \int_1^{\infty} x^2 e^{-x^3} dx$$

By substitution

$$\int_1^t x^2 e^{-x^3} dx$$

$$u = -x^3$$

$$du = -3x^2 dx$$

$$\frac{du}{-3} = x^2 dx$$

$$= -\frac{1}{3} \int e^u du = -\frac{1}{3} e^u + C$$

$$= -\frac{1}{3} e^{-x^3} \Big|_1^t$$

Part b:

$$\int_{-1}^8 \frac{1}{x^3} dx = \int_{-1}^0 \frac{1}{x^3} dx + \int_0^8 \frac{1}{x^3} dx$$

$$= \lim_{t \rightarrow 0^-} \int_{-1}^t \frac{1}{x^3} dx + \lim_{t \rightarrow 0^+} \int_t^8 \frac{1}{x^3} dx$$

$$= \lim_{t \rightarrow 0^-} \left[\frac{-1}{2t^2} + \frac{1}{2} \right] + \lim_{t \rightarrow 0^+} \left[\frac{-1}{2(64)} + \frac{1}{2t^2} \right]$$

$$= (-\infty + \frac{1}{2}) + (\frac{-1}{128} + \infty)$$

$$= -\infty + \infty \quad \text{Diverges} \quad \square$$

$$\int \frac{1}{x^3} dx = \int x^{-3} dx$$

$$= \frac{x^{-2}}{-2} + C$$

$$= \frac{-1}{2x^2} + C$$

Question 6:

Part a: Consider the following sequence $\left\{\frac{k^2-1}{k+k^2}\right\}_{k=2}^{\infty}$. Show that the sequence is increasing and determine whether the sequence is bounded above, and bounded below.

$$f(x) = \frac{x^2-1}{x+x^2} \Rightarrow f'(x) = \frac{(2x)(x+x^2) - (x^2-1)(1+2x)}{(x+x^2)^2}$$

$$f'(x) = \frac{2x^2+2x^3 - [x^2+2x^3-1-2x]}{(x+x^2)^2}$$

$$f'(x) = \frac{2x^2+2x^3-x^3-2x^3+1+2x}{(x+x^2)^2} = \frac{x^2+2x+1}{(x+x^2)^2} > 0$$

So, it's increasing ($a_{k+1} > a_k$) for $x \geq 1$

$$\frac{1}{2} \leq a_n = \frac{k^2-1}{k+k^2} \leq 1$$

Bounded below Bounded above

So, it's bounded

$$\lim_{k \rightarrow \infty} \frac{k^2-1}{k+k^2} = \frac{k^2}{k^2} = 1$$

Part b: Consider the following sequence $\left\{\frac{3^k}{(2k-1)!}\right\}_{k=1}^{\infty}$. Determine whether the sequence is increasing or decreasing or neither.

$$a_k = \frac{3^k}{(2k-1)!}$$

$$\frac{a_{k+1}}{a_k} = \frac{3^{k+1}}{(2k+1)!} \cdot \frac{(2k-1)!}{3^k} = \frac{3}{(2k+1)(2k)} < 1 \text{ for } k \geq 1$$

So, it's decreasing $a_{k+1} < a_k$.

Question 7: Determine if the series diverges or converges. Be sure to explain which test you use:

(i) Alternating Series $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$

(ii) $\lim_{n \rightarrow \infty} \frac{1}{\ln(n)} = \frac{1}{\ln(\infty)} = \frac{1}{\infty} = 0$

(iii) $f(x) = \frac{1}{\ln(x)} \Rightarrow f(x) = (\ln(x))^{-1}$
 $f'(x) = -(\ln(x))^{-2} \left(\frac{1}{x}\right) = -\frac{(\ln(x))^2}{x} < 0$
 $\Rightarrow f'(x) = -\frac{1}{x \ln^2 x} < 0$ for $n > 2$

So, it's convergent by Alternating Series Test. ✓

Question 8: How many terms of the following series:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$$

are needed to approximate the sum of the series with maximum error of $(0.5)10^{-4}$?

$$|E_{n+1}| \leq \frac{1}{n^4} \leq (0.5)(10^{-4})$$

$$\frac{1}{n^4} \leq \frac{0.5}{10^4}$$

$$\frac{1}{(n+1)^4} \leq \frac{0.5}{10^4}$$

$$(n+1)^4 \geq \frac{10^4}{0.5}$$

$$(n+1)^4 \geq 20,000$$

$$\sqrt[4]{(n+1)^4} \geq \sqrt[4]{20,000}$$

$$n+1 \geq \sqrt[4]{20,000}$$

$$n \geq \sqrt[4]{20,000} - 1$$

So, $n \approx 11.89 - 1$
 $n \approx 10.89$

✓

Question 9: Determine if the following series diverges or converges. Be sure to explain which test you use:

Part a:

$$\lim_{n \rightarrow \infty} \cos\left(\frac{1}{n}\right) = \cos\left(\frac{1}{\infty}\right) = \cos(0) = 1 \neq 0$$

So, it's divergent by N^{th} term test where the series oscillates between -1 and 1 . \square

Part b:

Compare with $\sum_{n=1}^{\infty} \frac{n}{n^{5/2}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ *leading terms* p-series $p=3/2 > 1$

$$\lim_{n \rightarrow \infty} \frac{n+2}{(n^{3/2}+n+1)(n+1)} \cdot \frac{n^{3/2}}{1} = \lim_{n \rightarrow \infty} \frac{n^{5/2}}{n^{5/2}} = 1 > 0$$

So, the series is convergent by limit Comparison Test. \square

Part c:

$\ln(n) \leq n^{\alpha}$ where α any positive number

$$\sum_{n=2}^{\infty} \frac{\ln(n)}{(n^{3/2}+1)}$$

$$\frac{\ln(n)}{n^{3/2}+1} \leq \frac{n^{0.1}}{n^{3/2}}$$

$$\frac{\ln(n)}{n^{3/2}+1} \leq \frac{1}{n^{1.4}} \Rightarrow \frac{\ln(n)}{n^{3/2}+1} \leq \frac{1}{n^{1.4}}$$

p-series $p=1.4 > 1$ So, it's convergent

by Direct Comparison Test. \square

Question 10: Determine the radius and interval of convergence for the following series:

By using ratio test:

$$\sum_{m=0}^{\infty} \frac{m^2}{m!} (x+1)^m$$

$$\lim_{m \rightarrow \infty} \left| \frac{(m+1)^2 \overset{(x+1)}{\cancel{(x+1)}^{m+1}}}{\underset{(m+1)}{\cancel{(m+1)!}}} \cdot \frac{\cancel{m!}}{m^2 \cancel{(x+1)}^m} \right| = \lim_{m \rightarrow \infty} |x+1| \frac{(m+1)^2}{m^2(m+1)} =$$

$$= |x+1| \lim_{m \rightarrow \infty} \frac{\overset{\text{leading terms}}{(m+1)^2}}{\underset{\text{leading terms}}{m^2(m+1)}} = |x+1| \lim_{m \rightarrow \infty} \frac{m^2}{m^3} = |x+1| \cdot 0 = 0$$

So, $R = \infty$ (Radius of Convergence)

Interval of Convergence (IC): $(-\infty, \infty)$



Question 11: Find the following using only Taylor's series:

$$\lim_{x \rightarrow 0} \left(\frac{\sin(x^3) - x^3}{x^9} \right)$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\lim_{x \rightarrow 0} \left[\frac{\left(\cancel{x^3} - \frac{x^9}{3!} + \frac{x^{15}}{15!} - \dots \right) \cancel{x^3}}{x^9} \right] = \lim_{x \rightarrow 0} \left(\frac{-1}{3!} + \frac{x^6}{15!} - \dots \right) =$$

$$= \boxed{\frac{-1}{6}}$$

Therefore, $\lim_{x \rightarrow 0} \left(\frac{\sin(x^3) - x^3}{x^9} \right) = \boxed{\frac{-1}{6}}$ \square

Question 12:

Part a: Change to rectangular coordinates and insert a graph below your result: $r = 6 \cos(\theta)$.

multiply both sides of $r = 6 \cos \theta$ by r , we obtain:

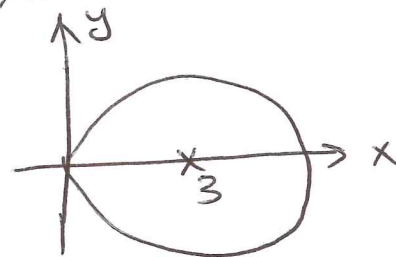
$$r^2 = 6r \cos \theta$$

$$x^2 + y^2 = 6x$$

$x^2 - 6x + y^2 = 0$ by completing the square, we have:

$$\underline{x^2 - 6x + 9} + y^2 = 0 + 9$$

$$(x-3)^2 + y^2 = 9 \quad \text{center} = (3, 0), \text{ radius} = 3$$



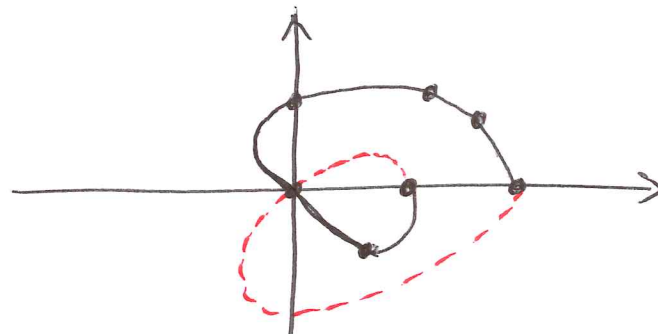
Part b: Sketch a graph for $r = 2 + 4 \cos(\theta)$ using symmetry tests.

Replace (r, θ) by $(r, -\theta)$

$$r = 2 + 4 \cos(-\theta)$$

$$r = 2 + 4 \cos(\theta)$$

Symmetric with respect to x, r axis



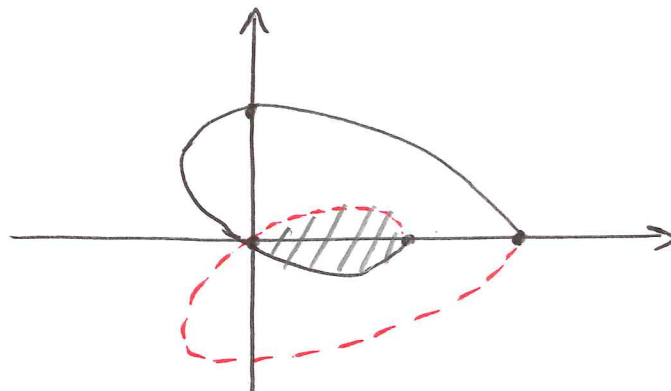
r	6	4.8	4	2	0	-0.8	-2
θ	0	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	π

Part c: Find the area inside the inner loop of the polar coordinates: $r = 1 + 2 \cos(\theta)$.

r	3	2	1	0	-1
θ	0	$\pi/3$	$\pi/2$	$2\pi/3$	π

$$A = 2 \left[\frac{1}{2} \int_{2\pi/3}^{\pi} (1 + 2 \cos \theta)^2 d\theta \right]$$

$$= \int_{2\pi/3}^{\pi} (1 + 4 \cos \theta + 4 \cos^2 \theta) d\theta$$



Question 13: Find the parametric equations for the line segment joining the points (1,2) and (4,7).

$$x = at + b$$

$$y = ct + d$$

$$\text{at } t=0 \Rightarrow C(1,2)$$

$$1 = b$$

$$2 = d$$

$$x = at + 1$$

$$y = ct + 2$$

$$4 = a + 1$$

$$7 = c + 2$$

$$a + 1 = 4$$

$$a = 3$$

$$c + 2 = 7$$

$$c = 5$$

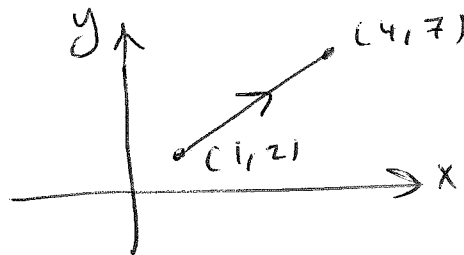
$$x = 3t + 1$$

$$y = 5t + 2$$

Check: $0 \leq t \leq 1$

$$t = \frac{x-1}{3}$$

$$y = \frac{5(x-1)}{3} + 2$$



Another Solution:

$$x = 1 + (4-1)t$$

$$y = 2 + (7-2)t$$

$$\left. \begin{array}{l} x = 1 + 3t \\ y = 2 + 5t \end{array} \right\}$$

$$0 \leq t \leq 1$$



Question 14:

Part a: Given $\vec{m} = \langle 2, 4 \rangle$ and $\vec{w} = \langle 3, 1 \rangle$. Find the following:

1- $\vec{m} + \vec{w}$

2- $\vec{m} - 2\vec{w}$

3- $\|5\vec{w} - 2\vec{m}\|$

Part ①: $\vec{m} + \vec{w} = \langle 2, 4 \rangle + \langle 3, 1 \rangle = \langle 5, 5 \rangle$

Part ②: $\vec{m} - 2\vec{w} = \langle 2, 4 \rangle - \langle 6, 2 \rangle = \langle -4, 2 \rangle$

Part ③: $5\vec{w} - 2\vec{m} = \langle 15, 5 \rangle - \langle 4, 8 \rangle = \langle 11, -3 \rangle$

$$\|5\vec{w} - 2\vec{m}\| = \sqrt{11^2 + 9} = \sqrt{130} \approx 11.40$$

□

Part b: Given $\vec{v} = \langle 6, 8 \rangle$. Find a unit vector \vec{u} in the same direction of \vec{v} .

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle 6, 8 \rangle}{\sqrt{36 + 64}} = \frac{\langle 6, 8 \rangle}{\sqrt{100}} = \frac{\langle 6, 8 \rangle}{10} =$$

$$\vec{u} = \left\langle \frac{6}{10}, \frac{8}{10} \right\rangle$$

$$\vec{u} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle \text{ unit vector}$$

□

Part c: Given $\vec{m} = \langle 3, 2, 0 \rangle$ and $\vec{w} = \langle -2, 4, 3 \rangle$. Find the following:

1- $\vec{m} \cdot \vec{w}$

2- The angle θ between \vec{m} and \vec{w} .

3- $\vec{m} \otimes \vec{w}$

Part ①: $\vec{m} \cdot \vec{w} = -6 + 8 + 0 = \boxed{2}$

Part ③: $\vec{m} \times \vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 0 \\ -2 & 4 & 3 \end{vmatrix}$

$= \langle 6, -9, 16 \rangle$

Part ②: $\vec{m} \cdot \vec{w} = 2$

$\|\vec{m}\| = \sqrt{9+4+0} = \sqrt{13}$

$\|\vec{w}\| = \sqrt{4+16+9} = \sqrt{29}$

$\cos \theta = \frac{2}{\sqrt{13} \sqrt{29}} \Rightarrow \cos \theta = \frac{2}{\sqrt{377}} \Rightarrow \theta = \cos^{-1}\left(\frac{2}{\sqrt{377}}\right)$

$\Rightarrow \therefore \theta \approx 84.088^\circ$



Part d: Find the distance from the point $A(2,0,1)$ to the line joining $B(1, -2, 2)$ and $C(3,0,2)$.

$\vec{BA} = \langle 1, 2, -1 \rangle$

$\vec{BC} = \langle 2, 2, 0 \rangle$

$\vec{BA} \times \vec{BC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ 2 & 2 & 0 \end{vmatrix} = \langle 2, -2, -2 \rangle$

$\|\vec{BA} \times \vec{BC}\| = \sqrt{4+4+4} = 2\sqrt{3}$

$\|\vec{BC}\| = \sqrt{4+4+0} = 2\sqrt{2}$

distance = $\frac{2\sqrt{3}}{2\sqrt{2}} = \sqrt{\frac{3}{2}}$



Good Luck in the Final Exam

Best of Luck

Mohammed K A Kaabar