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Practice With Volumes

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Review:

① General Slicing Method

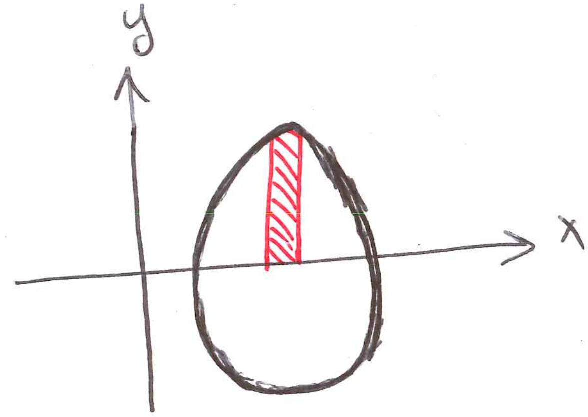
Suppose a solid object extends from $x=a$ to $x=b$ and the cross section of the solid perpendicular to the x-axis has an area given by a function A that is integrable on $[a,b]$. The volume of the solid is:

$$\boxed{V=\int_a^b A(x)\,dx}.$$

② Disk Method

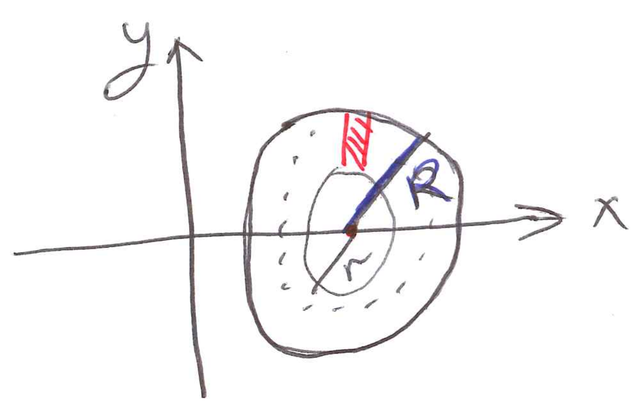
$$\text{Volume} = \pi r^2 t$$

radius (r), thickness (Slice) (t)



③ Washer Method

$$\text{Volume} = (\pi R^2 - \pi r^2)t$$

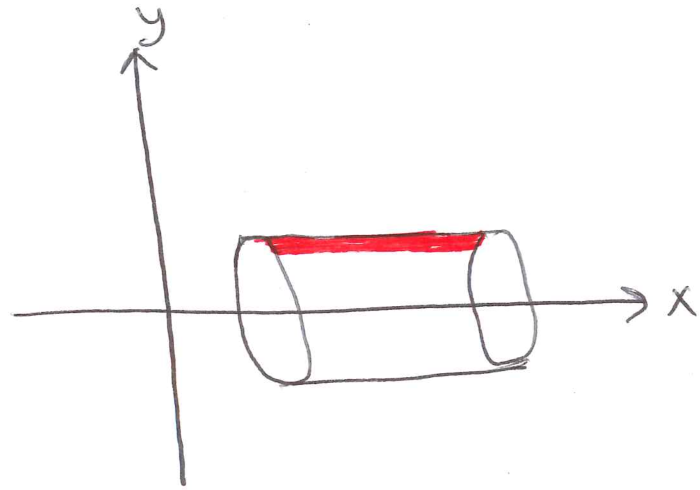


⟹ ①

④ Shell Method

$V = 2\pi r h t$

radius, height, thickness (slice)



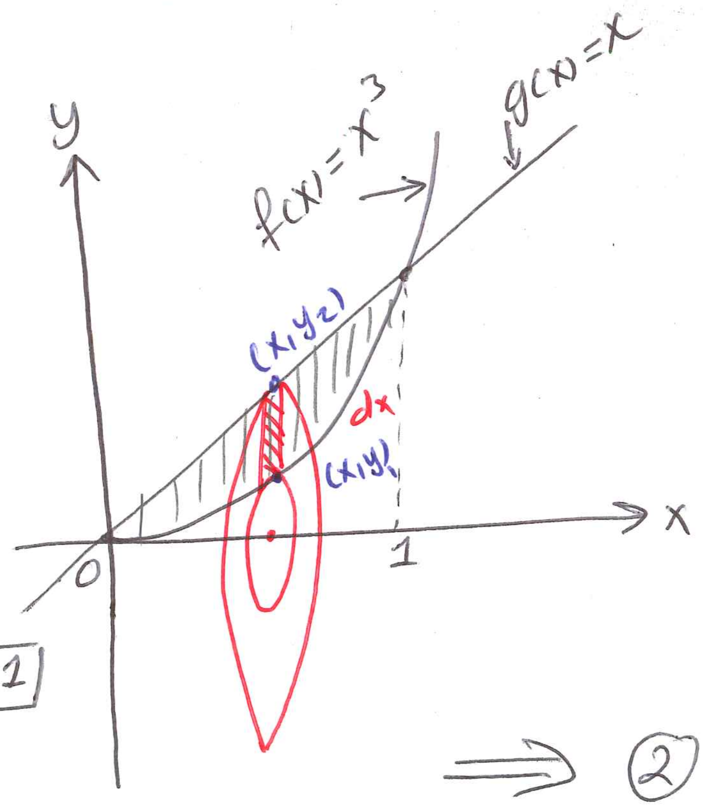
Example: (From Practice with Volumes Lab)

$f(x) = x^3$

$g(x) = x$

$x \geq 0$

Find the volume of the solid obtained by rotating the region bounded by $\boxed{f(x) = x^3}$ and $\boxed{g(x) = x}$ for $x \geq 0$ about x-axis?



Solution:

$x^3 = x$

$\frac{x^3}{x} = 1$

$x^2 = 1$

$x = \pm 1$ Since $x \geq 0$, we ignore $x = -1$. Thus, $\boxed{x = 0}$ or $\boxed{x = 1}$

$$V = (\pi R^2 - \pi r^2)\, t$$

$$V = \int_0^1 \left(\pi x^2 - \pi (x^3)^2\right) dx$$

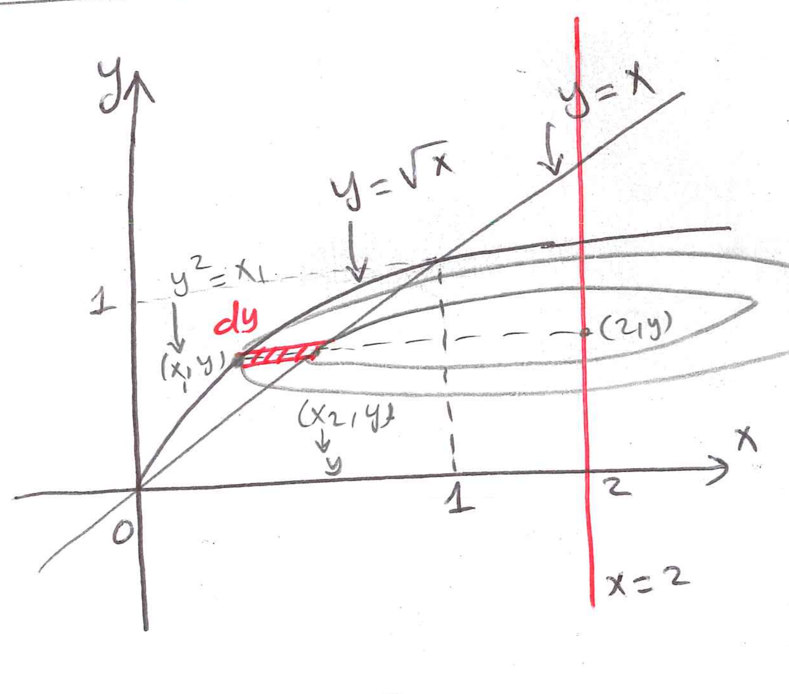
$$= \int_0^1 \pi (x^2 - x^6)\, dx = \pi \int_0^1 (x^2 - x^6)\, dx$$

$$= \pi \left[\frac{x^3}{3} - \frac{x^7}{7} \right] \Bigg|_0^1$$

$$= \boxed{\frac{\pi}{3} - \frac{\pi}{7}}$$

Lab Exercises:

Problem (A):



$$\sqrt{x} = x$$

$$x = x^2$$

$$1 = \frac{x^2}{x}$$

$$\boxed{1 = x}$$

$$V = (\pi R^2 - \pi r^2)\, t$$

$$V = \int_0^1 \left(\pi (2 - y^2)^2 - \pi (2 - y)^2\right) dy = \boxed{\frac{8\pi}{15}}$$

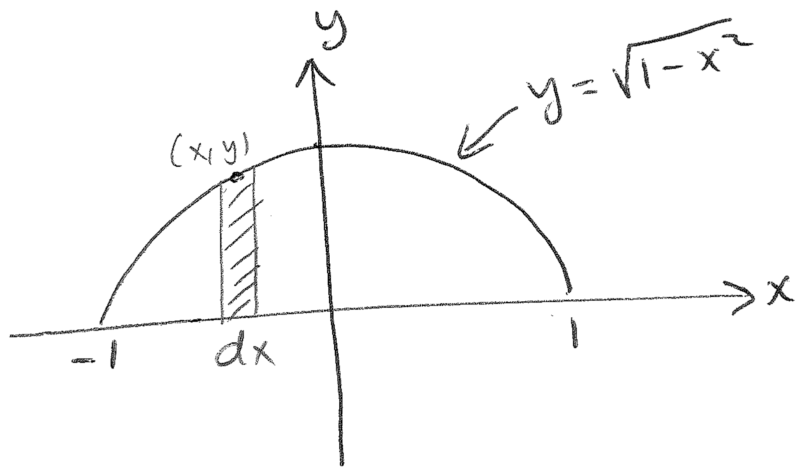
Problem (B):

$x^2+y^2=1$

Part (i): Semicircles

We know the area of semi-circles is $\frac{\pi r^2}{2}$

Since $x^2+y^2=1$, then $y^2=1-x^2 \Rightarrow \boxed{y=\sqrt{1-x^2}}$



$$V=\int_{-1}^{1} A(x)\,dx$$

$$=\int_{-1}^{1} \frac{\pi}{2}(y)^2\,dx$$

$$=\int_{-1}^{1} \frac{\pi}{2}\left(\sqrt{1-x^2}\right)^2 dx = \int_{-1}^{1} \frac{\pi}{2}(1-x^2)\,dx = \frac{\pi}{2}\int_{-1}^{1}(1-x^2)\,dx =$$

$$=\boxed{\frac{2\pi}{3}}$$

$\Longrightarrow$

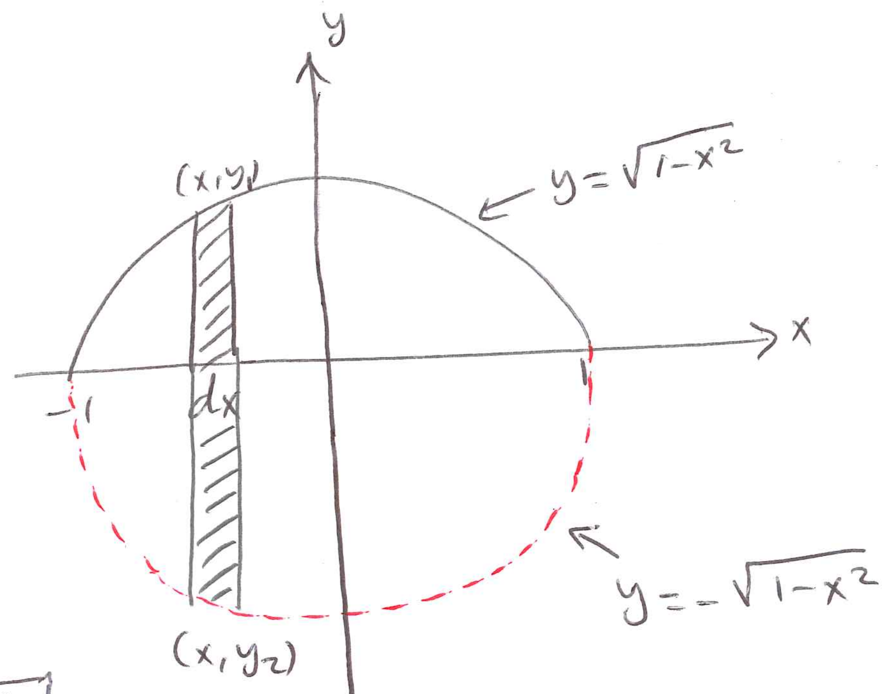
Part (ii): Squares

Area of square is L^2 where L is the length of the (cut) line.

$$V=\int_{-1}^{1} A(x)\,dx$$

$$=\int_{-1}^{1}\left(2\sqrt{1-x^2}\right)^2 dx$$

$$=\int_{-1}^{1}\left(4(1-x^2)\right) dx=\boxed{\frac{16}{3}}$$



$$L=y_1-y_2=\sqrt{1-x^2}+\sqrt{1-x^2}$$
$$=\boxed{2\sqrt{1-x^2}}$$

Part (iii):

Equilateral Triangle has a height of $\frac{b}{2}\sqrt{3}$ where b is the base

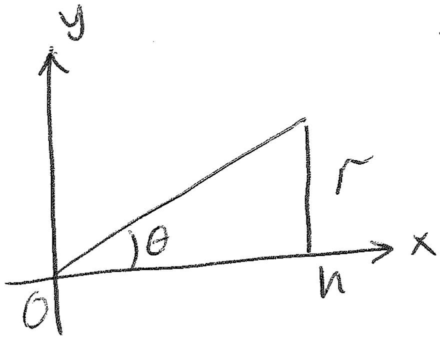
Area is $\boxed{\left(\frac{b}{2}\right)^2\sqrt{3}}$. In this example $\frac{b}{2}=\sqrt{1-x^2}$

So, Area becomes $\sqrt{3}\,(1-x^2) \Longrightarrow V=\int_{-1}^{1}\sqrt{3}(1-x^2)\,dx=\boxed{\frac{4\sqrt{3}}{3}}$

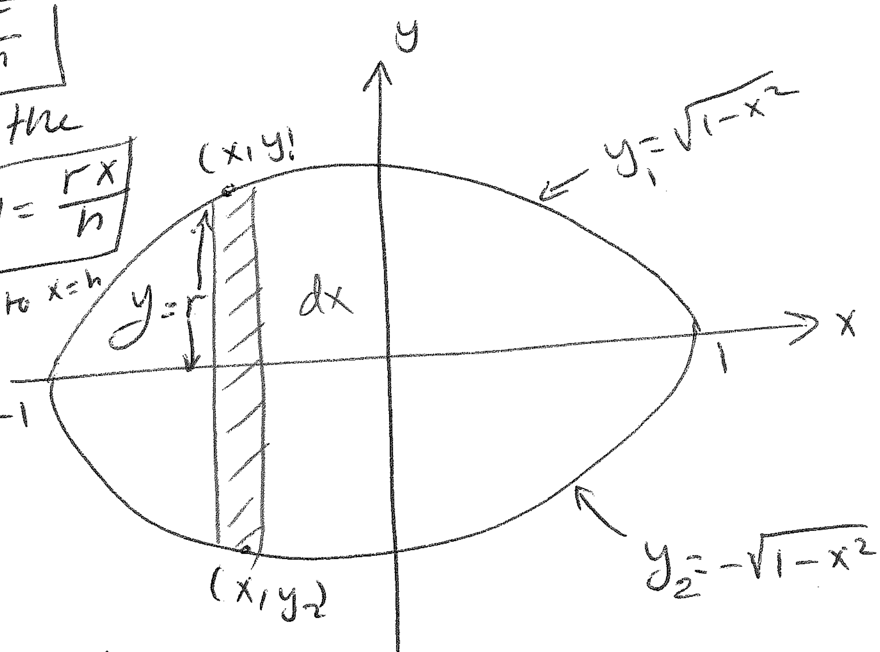
Problem (C):

$\boxed{\tan\theta = \frac{r}{h}}$

Equation of the line is $\boxed{y = \frac{rx}{h}}$ from $x=0$ to $x=h$



rotate



$$V = \int_a^b A(x)\,dx = \int_a^b \pi y^2\,dx$$

$$= \int_0^h \pi\left(\frac{rx}{h}\right)^2 dx$$

$$= \int_0^h \pi \frac{r^2x^2}{h^2}\,dx = \pi\left[\frac{r^2x^3}{3h^2}\right]\Bigg|_0^h$$

$$= \pi\left(\frac{r^2h^3}{3h^2} - 0\right)$$

$$= \boxed{\frac{\pi r^2 h}{3}}$$

thus, $\boxed{V = \frac{\pi r^2 h}{3}}$ by using the general slicing method.