Divergence and Curl   Math 273   Mohammed Kaabar

* Divergence and Curl are two operations on vector fields defined as follows:

Divergence: $\Rightarrow$ let $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$ and $F = \langle f, g, h \rangle$, then $\text{div}\, F = \nabla \cdot F = f_x + g_y + h_z$

$= \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$. (Note: the result is always scalar)

Curl: $\Rightarrow$ $\text{Curl}\, F = \vec{\nabla} \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix} =$

$$= \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\underline{i} - \left(\frac{\partial h}{\partial x} - \frac{\partial f}{\partial z}\right)\underline{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\underline{k}$$

Example: Suppose $F = \langle x^2 - y, 4z, x^2 \rangle$. Find div F and Curl F.

Solution: $\text{div}\, F = \frac{\partial}{\partial x}(x^2 - y) + \frac{\partial}{\partial y}(4z) + \frac{\partial}{\partial z}(x^2)$

$= 2x + 0 + 0 = 2x$

$\Longrightarrow$

$$\text{Curl } F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-y & 4z & x^2 \end{vmatrix}$$

*Note: A vector field (F) is called <u>conservative</u> if there is a function f for which $F = \nabla f = \langle f_x, f_y, f_z \rangle$ (Potential function)

$$= \left(\frac{\partial}{\partial y}(x^2) - \frac{\partial}{\partial z}(4z)\right)\underline{i} - \left(\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial z}(x^2-y)\right)\underline{j} +$$

$$\left(\frac{\partial}{\partial x}(4z) - \frac{\partial}{\partial y}(x^2-y)\right)\underline{k}$$

$$= (0-4)\underline{i} - (2x-0)\underline{j} + (0+1)\underline{k}$$

$$= -4\underline{i} - 2x\underline{j} + 1\underline{k}$$

$$= \langle -4, -2x, 1 \rangle$$

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*<u>theorem</u>: If F is <u>conservative</u>, then Curl $F = 0$.

<u>Notes</u>: ① If F is <u>NOT</u> conservative, then we <u>cannot</u> conclude Curl $F = 0$.

② If Curl $F \neq 0$, then it's for sure F is <u>NOT</u> conservative.

$\Longrightarrow$ ②

To explain that theorem, let take the following example that based on a mathematical concept known as "conditional statement":

Ex) If $\underbrace{\text{you have money}}_{\text{Call this part } (P)}$, $\underbrace{\text{you will buy a new car}}_{\text{Call this part } (Q)}$

So, If P, then Q

this means $P \xrightarrow{\text{implies}} Q$

Hence, the negation of the above statement is $\sim Q \longrightarrow \sim P$

$\sim$ means NOT (negate)

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Therefore, the notes of that theorem are based on the above conditional statement.

Example: Tell whether the following vector field is conservative or not if possible depending on Curl F.

Part (a): $F = \langle \cos(x-z), y^2, xz \rangle$

Solution: $\text{Curl}\,F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos(x-z) & y^2 & xz \end{vmatrix} = (-1-z)\underline{j} \neq 0$

Therefore, F is NOT conservative. $\boxed{\text{So, this cannot have a potential function}}$

Part (b): $F = \langle 2xz, 3z^2, x^2+6yz \rangle$

Solution: $\text{Curl}\,F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xz & 3z^2 & x^2+6yz \end{vmatrix}$

$= 0\underline{i} + 0\underline{j} + 0\underline{k} = \underline{0}$ ← zero vector

$= \langle 0, 0, 0 \rangle$

So, we don't know if F is conservative or NOT. $\boxtimes$

Notes about slides

For Figure 14.42, we notice the following:

Rotation is max when $\nabla \times F \,/\!/\, \underline{n}$.
$\hat{}$ parallel to

- $\nabla f \cdot n$ is the directional derivative in the direction $\underline{n}$.
- $(\nabla \times F) \cdot n$ is the directional spin (rotation) in the direction $\underline{n}$.

So, the paddle wheel spins fastest at $\theta = 0^\circ$ or $\theta = \pi^\circ$ because $(\nabla \times F) \cdot n = |\nabla \times F| \cos\theta$ where $|n| = 1$ and $\cos 0^\circ = \underline{\underline{1}}$, $\cos 180^\circ = \underline{\underline{-1}}$.

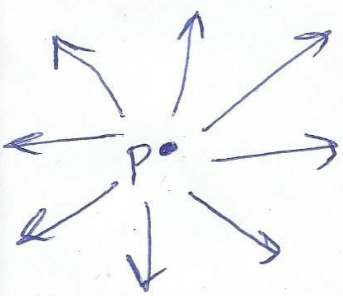
But if the axis of the paddle wheel is orthogonal to $\nabla \times F$, then the wheel will not spin because $(\nabla \times F) \cdot n = |\nabla \times F| \cos(90^\circ) = 0$ (No rotation).

$\Longrightarrow$ 5

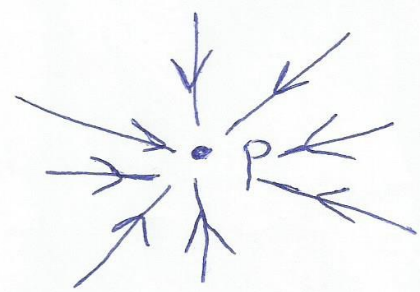
Divergence of a Vector Field (In Physics & Electromagnetics)

The divergence of $\vec{F}$ at a given point P is the outward flux per unit volume as the volume shrinks about P.

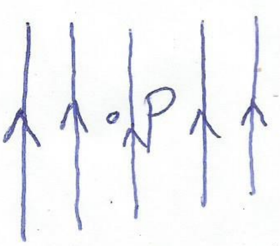
$$\Rightarrow \text{div}\,\vec{F} = \nabla \cdot \vec{F} = \lim_{\Delta V \to 0} \frac{\oint \vec{F} \cdot d\vec{s}}{\Delta V}$$



(+) divergence



(−) divergence



Zero divergence

Curl of Vector Field (In Physics and Electromagnetics)

The curl of $\vec{F}$ is an axial (rotational) vector where the magnitude is the max. circulation of $\vec{F}$ per unit area as the area tends to zero and whose direction is the normal direction of the area when the area is oriented so as to make the circulation maximum.

$$\Rightarrow \text{curl}\,\vec{F} = \nabla \times \vec{F} = \left[\lim_{\Delta s \to 0} \frac{\oint \vec{F} \cdot \vec{ds}}{\Delta s} \right] \cdot \underline{n}_{max}.$$