* Line Integrals   Math 273   Mohammed Kaabar

Definition: line integral is denoted by $\int_C f(x,y)\,ds$
where C is a curve (smooth curve).

This is similar to the integration in Calculus I over an interval $\int_a^b f(x)\,dx$ (look at Figure 14.16) →

Interpretation

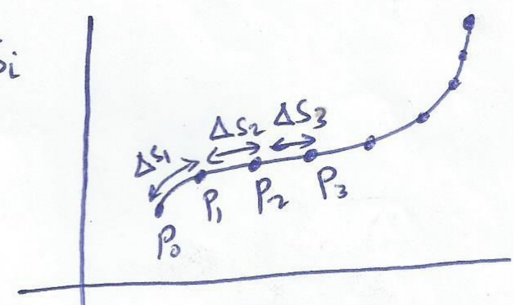
Assume that $f(x,y) \geq 0$, then $\int_C f(x,y)\,ds$ represents the area of one side of the "fence" whose base is C and whose height above the point (x,y) is $f(x,y)$.

* The Technique:

parametrize x and y: write x and y as functions of t say $x(t)$, $y(t)$ $a \leq t \leq b$

$$\int_C f(x,y)\,ds = \lim_{\substack{n\to\infty \\ \Delta\to 0}} \sum_{i=1}^{n} f(x_i, y_i)\Delta s_i$$

where $\Delta s_i = P_i - P_{i-1}$



length of arc:

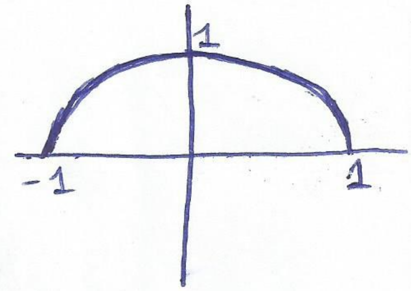
$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

Now, we obtain the 1st version for line integral:

Version I: $$\int_C f(x,y)\,ds = \int_a^b f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

Example: Evaluate $\int_C (2+x^2y)\,ds$ where C is the upper half of the unit circle: $x^2+y^2=1$

Solution:



$x(t)=\cos(t) \Longrightarrow \frac{dx}{dt}=-\sin(t)$

$y(t)=\sin(t) \Longrightarrow \frac{dy}{dt}=\cos(t)$

$0 \le t \le \pi$

$f(x(t),y(t))=2+\cos^2(t)\sin(t)$

$\sqrt{(x'(t))^2+(y'(t))^2}=\sqrt{(-\sin(t))^2+(\cos(t))^2}=\sqrt{1}=1$

Hence, $\int_C (2+x^2y)\,ds=\int_0^\pi (2+\cos^2(t)\sin(t))\,dt=$

$= 2t\Big|_0^\pi + \int_0^\pi \cos^2(t)\sin(t)\,dt$

Integration by substitution:
let $u=\cos(t)$
$du=-\boxed{\sin(t)\,dt}$
$\Rightarrow \sin(t)\,dt=-du$

$\Downarrow$

$-\int u^2\,du=\frac{-u^3}{3} \Rightarrow \frac{-\cos^3(t)}{3}$

$= 2t + -\frac{\cos^3(t)}{3}\Big|_0^\pi = \left[2\pi - \frac{\cos^3(\pi)}{3}\right]-\left[0-\frac{\cos^3(0)}{3}\right]$

$(-1)^3$, $1/3$

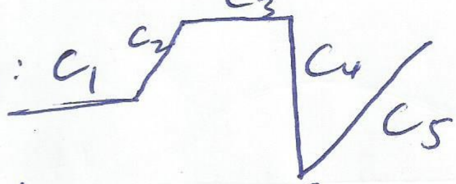
$=\left[2\pi+\frac{1}{3}\right]-\left[-\frac{1}{3}\right]=\boxed{2\pi+\frac{2}{3}}$

$\square$

* **Note:** If C is made up of a series of curves say:

$C = C_1 \cup C_2 \cup \cdots \cup C_n$

then, we obtain the following:

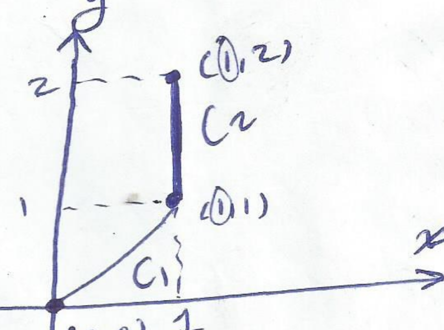


$$\int_C f(x,y)\,ds = \int_{C_1} f(x,y)\,ds + \int_{C_2} f(x,y)\,ds + \cdots + \int_{C_n} f(x,y)\,ds$$

Example: Evaluate $\int_C 2x\,ds$ where $C = C_1 \cup C_2$.

C_1 is a part of parabola $y = x^2$ from $(0,0)$ to $(1,1)$, and C_2 is the vertical line segment from $(1,1)$ to $(1,2)$.

Solution:



$$\int_C f(x,y)\,ds = \underbrace{\int_{C_1} f(x,y)\,ds}_{\text{parametrizing } C_1} + \underbrace{\int_{C_2} f(x,y)\,ds}_{\text{parametrizing } C_2}$$

$$\boxed{\begin{array}{l} x = x \Rightarrow \frac{dx}{dx} = 1 \\ y = x^2 \Rightarrow \frac{dy}{dx} = 2x \end{array}} \qquad \boxed{\begin{array}{l} y = y \Rightarrow \frac{dy}{dy} = 1 \\ x = 1 \Rightarrow \frac{dx}{dy} = 0 \end{array}}$$

$0 \le x \le 1$ $\qquad\qquad$ $1 \le y \le 2$

For C_1, we have: $\int_{C_1} f(x,y)\,ds = \int_0^1 2x\sqrt{1+(2x)^2}\,dx = \overbrace{\int_0^1 2x(1+4x^2)^{1/2}\,dx}^{\text{By subs.}}$

$$= \frac{1}{4}\cdot\frac{2}{3}(1+4x^2)^{3/2}\Big|_0^1 = \boxed{\frac{1}{6}\left(5\sqrt{5}-1\right)} \Longrightarrow$$

For C_2, we have: $\int_{C_2} f(x,y)\,ds = \int_1^2 2(1)\sqrt{0+1}\,dy = \int_1^2 2\,dy = 2$

Hence, $\int_C f(x,y)\,ds = \int_{C_1} f(x,y)\,ds + \int_{C_2} f(x,y)\,ds = \boxed{\frac{1}{6}(5\sqrt{5}-1)+2}$ ☑

---

* Line Integral (Version II):

$\int_C f(x,y)\,dx$ OR $\int_C f(x,y)\,dy$

$\Downarrow$

$$\boxed{\int_a^b f(x(t),y(t))\,x'(t)\,dt}$$

$\Downarrow$

$$\boxed{\int_a^b f(x(t),y(t))\,y'(t)\,dt}$$

Formula: For parametrizing the line segment starting from point (x_0,y_0) ending at (x_1,y_1):

$$\boxed{\begin{aligned} x(t) &= x_0 + (x_1 - x_0)t \\ y(t) &= y_0 + (y_1 - y_0)t \end{aligned}}$$

$0 \le t \le 1$

Example: Find the parametric equations for the line segment from $(\overset{x_0}{2},\overset{y_0}{1})$ to $(\overset{x_1}{4},\overset{y_1}{0})$.

Solution: $x(t) = 2 + (4-2)t \Rightarrow \boxed{x(t) = 2+2t}$

$y(t) = 1 + (0-1)t \Rightarrow \boxed{y(t) = 1-t}$

$0 \le t \le 1$ ☑

Example: Evaluate $\int_C y^2 dx + x\,dy$ where C is the line segment from $(\overset{x_0}{-5}, \overset{y_0}{-3})$ to $(\overset{x_1}{0}, \overset{y_1}{2})$

Solution:

$$\left.\begin{array}{l} x(t) = -5 + 5t \Rightarrow dx = 5\,dt \\ y(t) = -3 + 5t \Rightarrow dy = 5\,dt \end{array}\right\} \quad 0 \le t \le 1$$

$$\text{So, } \int_C y^2 dx + x\,dy = \int_0^1 (-3+5t)^2 (5\,dt) + \int_0^1 (-5+5t)(5\,dt)$$

$$= \text{Evaluate it} = \frac{160}{3}$$

$\square$

* Line Integral in Space (3d)

I) $$\int_C f(x,y)\,ds = \int_a^b f(x(t), y(t)) \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

$\Downarrow$

$$\text{In 3-D} \Rightarrow \int_C f(x,y,z)\,ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt$$

II) $$\int_C f(x,y)\,dx = \int_a^b f(x(t), y(t))\,x'(t)\,dt$$

$\Downarrow$

$$\text{In 3-D} \Rightarrow \int_C f(x,y,z)\,dx = \int_a^b f(x(t), y(t), z(t))\,x'(t)\,dt$$

Line Integral of Vector Fields (Version III)

Similar to the concept of "Work Done" in Calculus I. We can extend this concept to talk about the work done on an object by a force field:

$F(x,y,z)$ as the object moves along the curve C. We obtain:

Part 1: $W = \int_C \underbrace{F}_{\text{force field}} \underbrace{\cdot}_{\text{dot product}} \underbrace{T}_{\text{unit tangent vector}} \, ds$ (See Figure 14.19)

Part 2: "Very used and common"

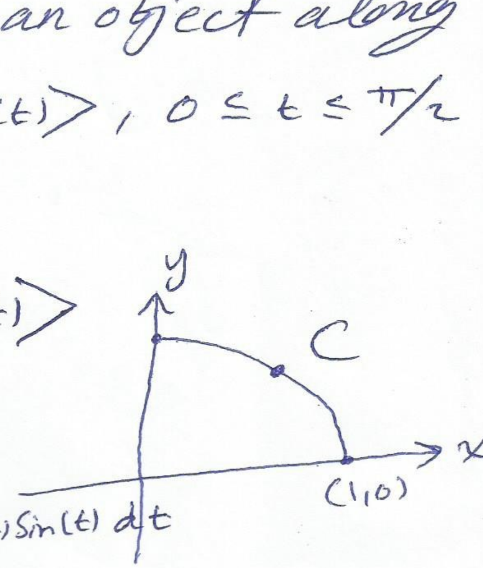
If C is given in a vector form say:

$C: \vec{r} = \langle x(t), y(t), z(t) \rangle$, then we have:

$$W = \int_C F \cdot dr \equiv \int_a^b F(r(t)) \cdot r'(t) \, dt$$

Example: Find the work done by force field $F(x,y) = \langle x^2, -xy \rangle$ in moving an object along a quarter circle, $r = \langle \cos(t), \sin(t) \rangle$, $0 \le t \le \pi/2$

Solution:



$F(r(t)) = \langle \cos^2(t), -\cos(t)\sin(t) \rangle$

$r'(t) = \langle -\sin(t), \cos(t) \rangle dt$

$F(r(t)) \cdot r'(t)\,dt = -\cos^2(t)\sin(t) - \cos^2(t)\sin(t)\,dt$

$= -2\cos^2(t)\sin(t)\,dt$

Hence, $W = \int_C F \cdot dr \equiv \int_a^b F(r(t)) \cdot r'(t)\,dt =$

$= \int_0^{\pi/2} -2\cos^2(t)\sin(t)\,dt = \underbrace{-2\int_0^{\pi/2} \cos^2(t)\sin(t)\,dt}_{\text{By substitution}}$

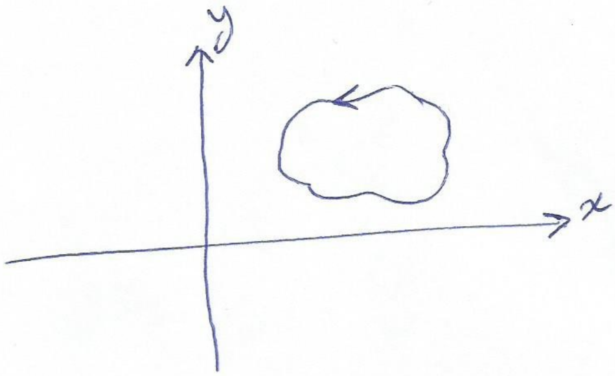
let $u = \cos(t)$

$du = -\boxed{\sin(t)\,dt} \Rightarrow \sin(t)\,dt = -du$

$\Rightarrow 2\int u^2\,du = 2\frac{u^3}{3} \Rightarrow 2\frac{\cos^3(t)}{3}\Big|_0^{\pi/2} = \frac{2}{3}\Big[\overbrace{\cos^3\left(\tfrac{\pi}{2}\right)}^{0} - \overbrace{\cos^3(0)}^{1}\Big]$

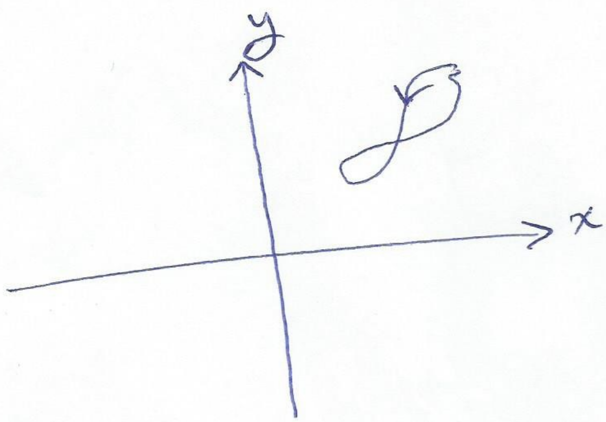
$= \frac{-2}{3}$ ☑

Different Types of Curves

① 

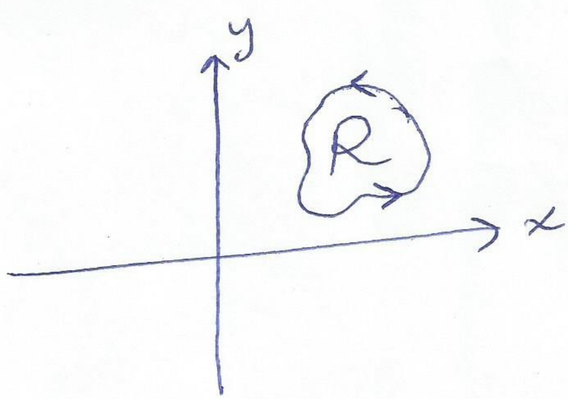
Simple Closed Curve

It does NOT have loops

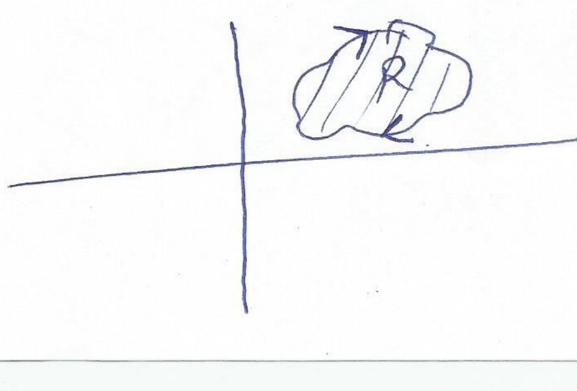
② 

Closed (Not Simple) Curve

It has loops

③ 

Positively-Oriented Curve

④ 

Negatively-Oriented Curve

Notation: $\oint_C F \cdot dr$ line integral over curve C

is "Simple Closed Positively-Oriented Curve"

* look at the definition of "Circulation" slide 18

* " " " " " "Flux" slide 22

* Explain the figures from 19 to 27.
