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Math 172 Lab
Fall 2015
Sections: 7 & 8

Thursday
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Review of Integration

$$x^2 \xrightarrow{\text{derivative}} 2x$$
$$2x \xrightarrow{\text{anti-derivative}} x^2$$

Def: Integration: - A function $F(x)$ is the anti-derivative of $f(x)$ if $F'(x) = f(x)$ for all x in the domain of f. Hence, the set of all anti-derivatives is called indefinite integral.

Example ①: Find $\int (x^2-1)^2 dx$.

Solution: To find $\int (x^2-1)^2 dx$, we need to simplify $(x^2-1)^2$ as $(x^2-1)^2 = x^4 - 2x^2 + 1$. Then, we have:

$$\int (x^2-1)^2 dx = \int (x^4 - 2x^2 + 1)\,dx = \boxed{\frac{x^5}{5} - \frac{2x^3}{3} + x + C}$$

Example ②: $\int x(x^2+1)\,dx \Leftarrow$ Evaluate it ?!

Solution: We use substitution method as follows:

let $u = x^2+1$

$du = 2\boxed{x\,dx} \Rightarrow x\,dx = \frac{du}{2}$

$\int u \frac{du}{2} = \frac{1}{2}\int u\,du =$

$\frac{1}{2}\frac{u^2}{2} + C = \frac{1}{4}u^2 + C$

$= \boxed{\frac{1}{4}(x^2+1)^2 + C}$

Example ③: Evaluate $\int \frac{x^4+1}{x^2}\,dx$.

Solution: $\int \frac{x^4+1}{x^2}\,dx = \int \left[\frac{x^4}{x^2} + \frac{1}{x^2}\right] dx = \int \left(x^2 + \frac{1}{x^2}\right) dx =$

$$= \int (x^2 + x^{-2})\,dx = \frac{x^3}{3} + \frac{x^{(-2+1)}}{(-2+1)} + C$$

$$= \frac{x^3}{3} + \frac{x^{-1}}{-1} + C$$

$$= \boxed{\frac{1}{3}x^3 - \frac{1}{x} + C}$$

Example ④: Find $\int \tan(x)\,dx$.

Solution: We know that $\tan(x) = \frac{\sin(x)}{\cos(x)}$. Then, we do the following: $\int \tan(x)\,dx = \int \left(\frac{\sin(x)}{\cos(x)}\right) dx =$

$$= -\int \frac{-\sin(x)}{\cos(x)}\,dx = \boxed{-\ln|\cos(x)| + C}$$

Note:

$$(\ln \square)' = \frac{\square'}{\square}$$

$$(e^{\square})' = e^{\square} \cdot \square'$$

Example ⑤: Find $\int \frac{1}{x^2+6x+13}\,dx$

Solution: We use "Completing the square" method as follows: $x^2 + 6x + 13 = \boxed{\underbrace{x^2 + 6x + 9}_{(x+3)^2}} + 4 = \boxed{(x+3)^2 + 4} \Longrightarrow$

$\frac{6}{2} = (3)^2 = 9$

Continue

So, $\int \frac{1}{x^2+6x+13}\,dx = \int \frac{1}{(x+3)^2+4} = \frac{1}{4}\int \frac{1}{\frac{(x+3)^2}{4}+\frac{4}{4}}\,dx =$

$= \frac{1}{4}\int \frac{1}{\frac{(x+3)^2}{4}+1}\,dx = \frac{1}{4}\int \frac{1}{\left(\frac{x+3}{2}\right)^2+1}\,dx$

Now, we use "substitution method" as follows:

let $u = \frac{x+3}{2} = \frac{x}{2}+\frac{3}{2}$

$du = \frac{1}{2}\,dx \Rightarrow \boxed{dx = 2\,du}$

$\Rightarrow \frac{1}{4}\int \frac{1}{\left(\frac{x+3}{2}\right)^2+1}\,dx = \frac{1}{4}\int \frac{1}{u^2+1}\cdot 2\,du = \frac{2}{4}\int \frac{1}{u^2+1}\,du =$

$= \frac{1}{2}\int \frac{1}{u^2+1}\,du = \frac{1}{2}\tan^{-1}(u) + C$

$= \boxed{\frac{1}{2}\tan^{-1}\left(\frac{x+3}{2}\right) + C}$

Note:

We know that

$\int \frac{1}{1+x^2}\,dx = \boxed{\tan^{-1}(x) + C}$

* The following is a list of some common examples of integration in Calculus II:

- $\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x) + C$
- $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x) + C$

Continue $\Longrightarrow$

- $\int \frac{-1}{\sqrt{1-x^2}}\, dx = \cos^{-1}(x) + C$

- $\int e^{2x} dx = \frac{e^{2x}}{2} + C$

- $\int \sin(kx) dx = \frac{-\cos(kx)}{k} + C$

Example ⑥: Evaluate $\int \underbrace{x}_{u}\, \underbrace{e^x dx}_{dv}$.

Solution: We use "integration by parts" as follows:

Remember: $\int u\, dv = uv - \int v\, du$

Note: Please ignore this example for now. We will discuss it in the next coming weeks.

let $u = x$ $\quad dv = e^x dx$

$du = dx$ $\quad v = \int e^x dx = e^x$

$$\int x e^x dx = x e^x - \int e^x dx$$

$$= \boxed{x e^x - e^x + C}$$

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* Riemann Sums: -

$\sum$ it's called "Sigma" means summation (Sum).

Example ⑦: $1 + 2 + 3 + \cdots + 50 = \sum_{i=1}^{50} i$

$\Longrightarrow$

Example ⑧: $\sum_{i=1}^{5}(2i+1) = 3+5+7+9+11 = 35$

Example ⑨: $\sum_{i=1}^{10} 1 = 1+1+1+ - - - +1 = 10$

* Properties of Σ (Sigma)

ⓐ $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$

ⓑ $\sum_{i=1}^{n}(a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$

* Theorem about Σ:

① $\sum_{i=1}^{n} i = \frac{(n)(n+1)}{2}$

② $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

③ $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$

Proof ①:

$$\begin{array}{l} 1+2+3+ - - - +(n-2)+(n-1)+n \\ n+(n-1)+(n-2)+ - \cdot + 3+2+1 \\ \hline \Rightarrow (n+1)+(n+1)+(n+1)+ - - \cdot +(n+1) \end{array}$$

$$= \frac{(n+1)n}{2}$$

Example (10): Evaluate $\sum_{i=1}^{10}(2i^2+3i-7)$.

Solution:

$$\sum_{i=1}^{10}(2i^2+3i-7)=\sum_{i=1}^{10}2i^2+\sum_{i=1}^{10}3i-\sum_{i=1}^{10}7$$

$$=2\sum_{i=1}^{10}i^2+3\sum_{i=1}^{10}i-\sum_{i=1}^{10}7$$

$$=2\,\frac{\overset{10}{n}(\overset{10}{n}+1)(2\overset{10}{n}+1)}{6}+3\,\frac{\overset{10}{n}(\overset{10}{n}+1)}{2}-70$$

$$=\frac{2(10)(11)(21)}{6}+\frac{3(10)(11)}{2}-70$$

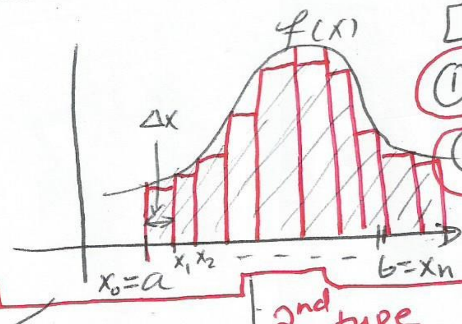
$$=770+155-70$$

$$=\boxed{855}$$

* Riemann Sums:

Three Types:

$[a,b]$ $n \Rightarrow$ number of subintervals



① $\Delta x=\dfrac{b-a}{n}$

② $x_i=x_0+i\,\Delta x$

③ $R_n=\sum_{i=1}^{n}f(x_i)\Delta x$ Riemann Sum

④ Area $=\lim_{n\to\infty}R_n$

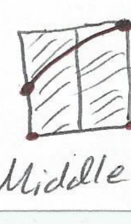
1st type

left Sum

2nd type

Right Sum

3rd type



Middle Sum

Theorem: A function f defined on $[a,b]$ is integrable on $[a,b]$ if $\lim_{n\to\infty} R_n = \lim_{n\to\infty}\left[\sum_{i=1}^{n} f(x_i)\,\Delta x\right]$ exists and is unique over all partitions of $[a,b]$ and all choices of x_i on a partition. This is known as definite integral of f from a to b as follows:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x .$$

Example (11): Calculate the area bounded by the graph of $f(x)=x^2$ and x-axis, between $x=0$ and $x=8$ for 4 sub-intervals?

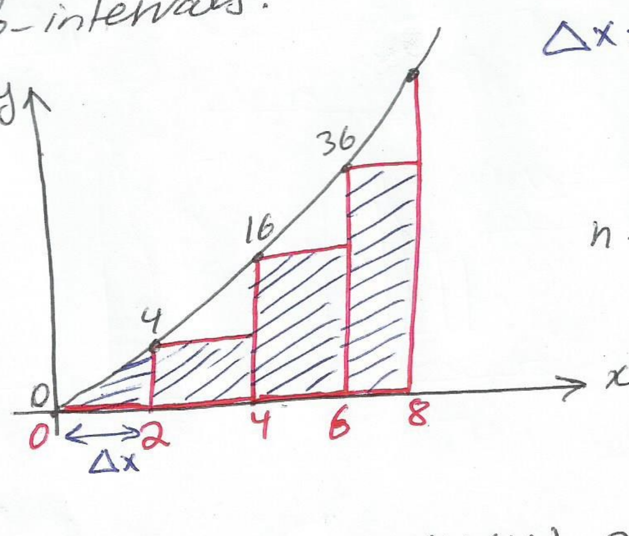
$\Delta x = \frac{b-a}{n} = \frac{8-0}{4} = 2$

$[0,8]$

n - rectangles

Solution:

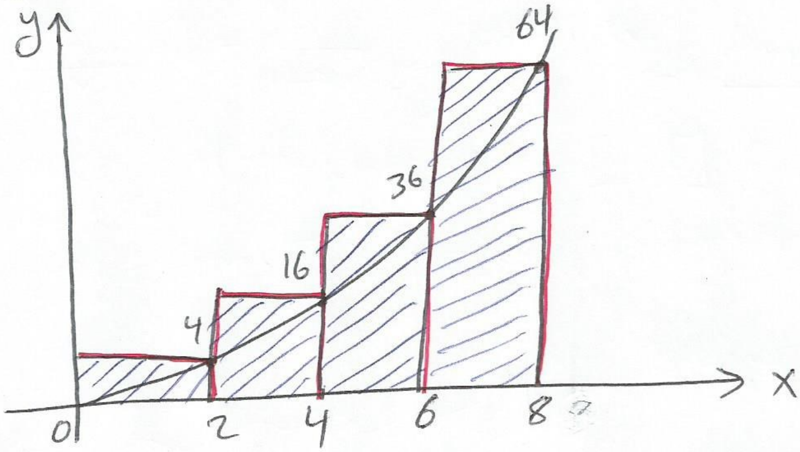
I. Left Sum:



Riemann Sum for 4 sub-intervals

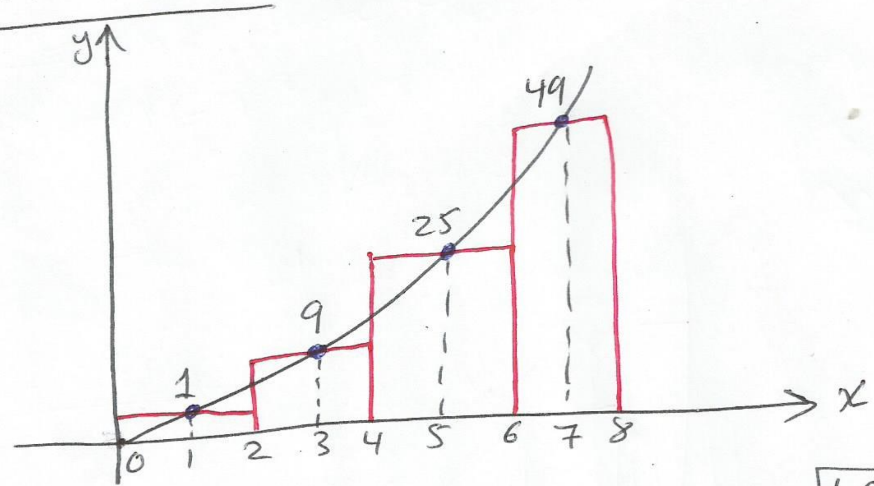
$$R_4 = \sum_{i=1}^{4} f(x_i)\,\Delta x = (2)(0)+(2)(4)+(2)(16)+2(36) = \boxed{110}$$

II. Right Sum



$R_4 = (2)(4) + (2)(16) + (2)(36) + (2)(64) = \boxed{236}$

III. Middle Sum



$R_4 = (2)(1) + (2)(9) + (2)(25) + (2)(49) = \boxed{168}$

Now, let's compare them with actual area as follows:

$\Longrightarrow$

$$\text{Area} = \int_0^8 x^2\,dx = \frac{x^3}{3}\Big|_0^8 \approx 170.666667$$

As we can notice that the middle sum gave the best estimate comparing with other types of Riemann Sums.