Mohammed KA Kaabar | Taylor Approximation | Nov. 5th, 2015

* **Theorem**: Suppose that $f(x)$ has $(n+1)$ derivatives on the interval $(c-r, c+r)$, $r>0$. Then, for $x \in (c-r, c+r)$, $f(x) \approx P_n$ and the error is approximating $f(x)$ by $P_n(x)$ is:

$$|E(x)| = |f(x) - P_n(x)| = \left| \frac{f^{(n+1)}(c)\,(x-a)^{n+1}}{(n+1)!} \right|$$ where c is between a and x.

Example ①: Given $f(x) = \ln(x)$ centered at $a=1$. Find $\ln(1.1)$ using $P_4(x)$. Then, use the appropriate taylor's series to approximate $\ln(1.1)$ accurate to 10^{-5}.

Solution:

$f(x) = \ln(x) \longrightarrow f(1) = \ln(1) = 0$

$f'(x) = \frac{1}{x} = x^{-1} \longrightarrow f'(1) = 1$

$f''(x) = -x^{-2} \longrightarrow f''(1) = -1$

$f'''(x) = 2x^{-3} \longrightarrow f'''(1) = 2$

$f^{(4)}(x) = -6x^{-4} \longrightarrow f^{(4)}(1) = -6$

$$\ln(x) \approx \underbrace{\boxed{(x-1) - \frac{(x-1)^2}{2!} + 2\frac{(x-1)^3}{3!} - \frac{6(x-1)^4}{4!}}}_{P_4(x)}$$

$$\ln(x) \approx (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4}$$

$$\ln(1.1) \approx 0.1 - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4} \Rightarrow \boxed{\ln(1.1) \approx 0.024}$$

$\Rightarrow$ ①

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$$|E| = \left| \frac{f^{(5)}(c)(x-1)^5}{5!} \right| \Longrightarrow$$ Now, $f^{(5)}(c)$ can be found as follows

$$f^{(5)}(x) = 24x^{-5} \Longrightarrow \boxed{f^{(5)}(c) = 24c^{-5}}$$

So, $|E| = \left| \frac{24}{c^5} \cdot \frac{(x-1)^5}{5!} \right|$ where c is between 1 and $x \in (1, 1.1)$

Hence, $$\left| \frac{1}{5c^5} \cdot (1.1 - 1)^5 \right| = \frac{(0.1)^5}{5c^5} < \frac{(0.1)^5}{5} < 10^{-5}. \quad \blacksquare$$