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Trigonometric Substitution & Partial Fractions

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* Trigonometric Techniques:

Suppose that you have the following integration:

$$\boxed{\int \sin^n x \cos^m x \, dx}$$ where n and m are positive integers.

I. m or n are positive $\boxed{\text{odd}}$ integers

We know
$\sin^2 x + \cos^2 x = 1$
$\Rightarrow \sin^2 x = 1 - \cos^2 x$

Ex) $\int \cos^4 x \sin^3 x \, dx$

$$\Rightarrow \int \cos^4 x \sin^3 x \, dx = \int \cos^4 x \sin^2 x \sin x \, dx$$
$$= \int \cos^4 x (1 - \cos^2 x) \sin x \, dx$$

Now, let $u = \cos(x)$
$du = -\sin(x) dx \Rightarrow \sin x \, dx = -du$

$$\text{So, } -\int u^4 (1 - u^2) du = -\int (u^4 - u^6) du = -\frac{u^5}{5} + \frac{u^7}{7} + C$$
$$= \boxed{-\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7} + C}$$

$\Longrightarrow$

II. m and n are positive even integers

Ex $\int \sin^2 x\,dx$

We know the half-angle formula

$$\sin^2 x = \frac{1-\cos(2x)}{2}$$

$$\cos^2 x = \frac{1+\cos(2x)}{2}$$

$$\Rightarrow \int \sin^2 x\,dx = \int \left(\frac{1-\cos(2x)}{2}\right)dx$$

$$= \int \left[\frac{1}{2} - \frac{1}{2}\cos(2x)\right]dx$$

$$= \frac{1}{2}x - \frac{1}{2}\,\frac{\sin(2x)}{2} + C$$

$$= \boxed{\frac{1}{2}x - \frac{1}{4}\sin(2x) + C}$$

III. m is only odd positive integer $\Rightarrow \int \tan^m x \sec^n x\,dx$

Ex $\int \tan^3 x \sec^3 x\,dx$

Remember

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

$$\Rightarrow \int \tan^3 x \sec^3 x\,dx = \int \underbrace{\tan^2 x}_{\sec^2 x - 1} \sec^2 x \sec x \tan x\,dx$$

$u = \sec x$

$du = \sec x \tan x\,dx$

$$\Rightarrow \int (u^2-1)u^2\,du$$

$$\Rightarrow \int (u^4 - u^2)\,du = \frac{u^5}{5} - \frac{u^3}{3} + C$$

$$= \boxed{\frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C}$$

IV. n is an even integer $\Rightarrow \int \tan^m x \sec^n x \, dx$

Ex $\int \tan^3 x \sec^4 x \, dx$

$$\Rightarrow \int \tan^3 x \underbrace{\boxed{\sec^2 x}}_{=\boxed{1+\tan^2 x}} \sec^2 x \, dx$$

$u = \tan(x)$

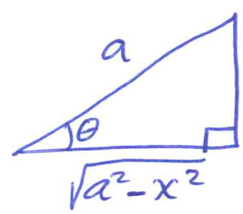
$du = \sec^2 x \, dx$

$$\Rightarrow \int u^3 (1+u^2) \, du = \int (u^3 + u^5) \, du = \frac{u^4}{4} + \frac{u^6}{6} + C$$

$$= \boxed{\frac{\tan^4 x}{4} + \frac{\tan^6 x}{6} + C}$$

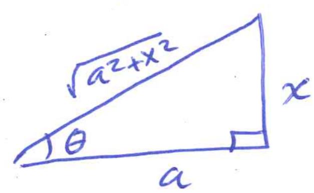
* Trigonometric Substitution

① $\sqrt{a^2 - x^2}$ where $x = a\sin\theta$



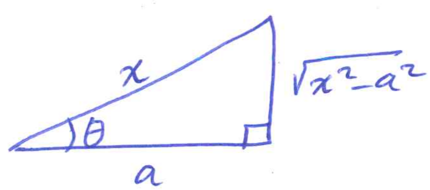
$\sin\theta = \frac{x}{a}$

② $\sqrt{a^2 + x^2}$ where $x = a\tan\theta$



$\tan\theta = \frac{x}{a}$

③ $\sqrt{x^2 - a^2}$ where $x = a\sec\theta$



$\sec\theta = \frac{x}{a}$

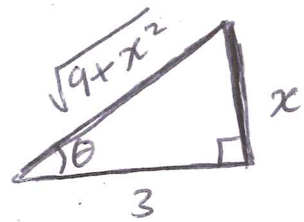
$\cos\theta = \frac{a}{x}$

where $\sec\theta = \frac{1}{\cos\theta}$

Ex $\int \frac{1}{\sqrt{9 + x^2}} \, dx$

$x = 3\tan\theta$

$dx = 3\sec^2\theta \, d\theta$



$\Longrightarrow$

$$\sqrt{9 + 9\tan^2\theta} = \sqrt{9(1+\tan^2\theta)}$$

$$= 3\sqrt{1+\tan^2\theta} = 3\sqrt{\sec^2\theta}$$

$$= \boxed{3\sec\theta}$$

$$\Rightarrow \int \frac{1}{\cancel{3}\cancel{\sec\theta}} \cdot \cancel{3}\sec^2\theta \, d\theta = \int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{\sqrt{9+x^2}}{3} + \frac{x}{3}\right| + C$$

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* Partial Fractions

$$\frac{1}{3} + \frac{1}{5} = \frac{5+3}{15} = \frac{8}{15}$$

$$\frac{1}{3} + \frac{1}{5} \xrightarrow{\text{LCD}} \frac{8}{15}$$

$$\frac{8}{15} \xrightarrow{\text{Partial Fractions}} \frac{1}{3} + \frac{1}{5}$$

If $\boxed{\deg(\text{numerator}) < \deg(\text{denominator})}$ and denominator can be factored, then we can use Partial Fractions.

degree

degree

$\boxed{\text{We know } x^3 - x = x(x^2-1) = x(x-1)(x+1)}$

$\boxed{\text{Ex}}$ $\displaystyle\int \frac{3x^2 - 7x - 2}{x^3 - x}\, dx$

By Partial Fractions: $\dfrac{3x^2 - 7x - 2}{x^3 - x} = \dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{C}{x+1}$

$$\frac{3x^2 - 7x - 2}{x(x-1)(x+1)} = \frac{A(x-1)(x+1) + Bx(x+1) + Cx(x-1)}{x(x-1)(x+1)}$$

$$3x^2 - 7x - 2 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$$

$\underline{x=0}$: $\cancel{-}2 = \cancel{-}A \Rightarrow \boxed{A = 2}$

$\Longrightarrow$

$\Rightarrow \underline{x=1}: -6 = 2B \Rightarrow \boxed{B=-3}$

$\boxed{\ln x^2 = 2\ln x}$

$\Rightarrow \underline{x=-1}: 8 = 2C \Rightarrow \boxed{C=4}$

Thus, $\int \frac{3x^2-7x-2}{x^3-x}\,dx = \int \left[\frac{2}{x} - \frac{3}{x-1} + \frac{4}{x+1}\right]dx$

$= \int \frac{2}{x}\,dx - 3\int \frac{1}{x-1}\,dx + 4\int \frac{1}{x+1}\,dx$

$= \boxed{2\ln|x| - 3\ln|x-1| + 4\ln|x+1| + C}$

$= \boxed{\ln\left|\frac{x^2(x+1)^4}{(x-1)^3}\right| + C}$

This is the simplest form

Now, to check our solution, let's use a method called "Cover Method" as follows: $\Rightarrow$ Ex| $\int \frac{3x^2-7x-2}{x^3-x}\,dx = \int \frac{3x^2-7x-2}{x(x-1)(x+1)}\,dx$

① Cover the original.

② Substitute $\boxed{x=0}$ in $\frac{3x^2-7x-2}{\boxed{}(x-1)(x+1)} \Rightarrow A = \frac{-2}{(-1)(1)} = \boxed{2}$

here we already covered x.

③ Substitute $\boxed{x=1}$ in $\frac{3x^2-7x-2}{\boxed{}x(x+1)} \Rightarrow B = \frac{3-7-2}{1(1+1)} = \frac{-6}{2} = \boxed{-3}$

here we already covered $(x-1)$

④ Substitute $\boxed{x=-1}$ in $\frac{3x^2-7x-2}{\boxed{}x(x-1)} \Rightarrow C = \frac{3+7-2}{-1(-1-1)} = \frac{8}{2} = \boxed{4}$

here we already covered $(x+1)$ $\Longrightarrow$

Therefore, we got $\boxed{A=2}$, $\boxed{B=-3}$, and $\boxed{C=4}$ which is the same result we got previously from the traditional way of partial fractions. You may use still use this method in exams and quizzes to check your solution using the traditional way of Partial fractions.