Mohammed Kaabar Taylor Polynomials Nov. 3rd, 2015

*Definition: Power Series

A power series about $x=0$ has the form:

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

A power series about $x=a$ has the form:

$$\sum_{n=0}^{\infty} a_n (x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots$$

Example ①: Investigate: $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$

Solution: $1 + \frac{1}{2} + \frac{1}{4} + \cdots$

$a = 1$

$r = \frac{1}{2} < 1$ convergent

$$S_n = \frac{a}{1-r} = \frac{1}{1-1/2} = \frac{1}{1/2} = 2$$

☒

*three cases about the radius of convergence and interval of convergence:

① Convergent on an interval

$a < x < b$

$a \leq x < b$

$a < x \leq b$

$a \leq x \leq b$

$$\boxed{\text{Radius of Convergence} = R = a > 0}$$

⟹

② Convergent always on $-\infty < x < \infty$

Radius of Convergent $R = \infty$

③ Convergent only at $x = 0$

Radius of Convergent $R = 0$

Example ②: Investigate: $\sum_{n=0}^{\infty} n!x^n$

Solution: $\sum_{n=0}^{\infty} n!x^n = 1 + x + 2!x^2 + 3!x^3 + \cdots$

$$\lim_{n\to\infty} \left| \frac{\cancel{(n+1)!}^{(n+1)} \cdot x^{\cancel{(n+1)}\ x^1}}{\cancel{n!}\ \cancel{x^n}} \right|$$ Ratio Test

$\Rightarrow \lim_{n\to\infty} (n+1)|x| = \infty > 1$ diverges unless $x = 0$

So, it's convergent only at $x = 0$, $R = 0$. ▨

*Definition: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!} = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2!}$
$+ \frac{f'''(a)}{3!}(x-a)^3 + \cdots$ Taylor's expansion about $x = a$.

If $a = 0$, it's called "Maclaurin Series".

$\Longrightarrow$

 Mohammed Kaabar

Example ③: Find the 4^{th} degree Taylor Polynomial for $f(x) = \frac{1}{x}$ centered at $x = 1$.

Solution:

$f(x) = \frac{1}{x} = x^{-1}$ $\qquad f(1) = 1$

$f'(x) = -x^{-2}$ $\qquad f'(1) = -1$

$f''(x) = 2x^{-3}$ $\qquad f''(1) = 2$

$f'''(x) = -6x^{-4} = -3!\,x^{-4}$ $\qquad f'''(1) = -3!$

$f^{(4)}(x) = 4!\,x^{-5}$ $\qquad f^{(4)}(1) = 4!$

$$\frac{1}{x} = 1 - (x-1) + \frac{\cancel{2}(x-1)^2}{\cancel{2!}} - \frac{\cancel{3!}(x-1)^3}{\cancel{3!}} + \frac{\cancel{4!}(x-1)^4}{\cancel{4!}} - - - - =$$

$$\frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 - - - - -$$

$$= \sum_{n=0}^{\infty} (-1)^n (x-1)^n \quad \blacksquare$$