Name: Mohammed K A Kaabar

Section: - Solution -

Telescoping and Geometric Series Activity

Introduction: In this lab we will evaluate infinite geometric and telescoping series. Non-trivial geometric series are series that can be written in the form $\boxed{\sum_{k=0}^{\infty} ar^k}$ where a and r are non-zero real numbers. Recall that if $|r| < 1$ the geometric series sums to $\frac{a}{1-r}$, and if $|r| \geq 1$ then the series diverges. Very loosely speaking, telescoping series are series for which you can define the n^{th} partial sum of the series, denoted S_n, in such a way that $\boxed{\lim_{n\to\infty} S_n}$ can be evaluated exactly, and whose partial sums eventually have only a fixed number of terms after cancellation.

Ex: 1.) Find the sum of the sequence.

$$\sum_{k=2}^{\infty} \frac{1}{k^2-1}$$

Solution:

Using the technique of partial fractions we find that

$$\frac{1}{k^2-1} = \frac{1}{(k+1)(k-1)} = \frac{1}{2k-2} - \frac{1}{2k+2}$$

$$= \frac{1}{2}\left[\frac{1}{k-1} - \frac{1}{k+1}\right] = a_k$$

and thus we have that

$$\sum_{k=2}^{\infty} \frac{1}{k^2-1} = \sum_{k=2}^{\infty} \frac{1}{(k+1)(k-1)} = \frac{1}{2}\sum_{k=2}^{\infty}\left[\frac{1}{k-1} - \frac{1}{k+1}\right]$$

using the following telescoping technique for $n > 2$ we can try to find a pattern for S_n:

$$S_2 = a_2 = \frac{1}{2}\left(1 - \frac{1}{3}\right)$$

$$S_3 = S_2 + a_3 = \frac{1}{2}\left(1 - \frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{1}{2}\left(\frac{3}{2} - \frac{1}{3} - \frac{1}{4}\right)$$

$$S_4 = S_3 + a_4 = \frac{1}{2}\left(\frac{3}{2} - \frac{1}{3} - \frac{1}{4}\right) + \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{1}{2}\left(\frac{3}{2} - \frac{1}{4} - \frac{1}{5}\right)$$

$$S_5 = \frac{1}{2}\left(\frac{3}{2} - \frac{1}{4} - \frac{1}{5}\right) + \frac{1}{2}\left(\frac{1}{4} - \frac{1}{6}\right) = \frac{1}{2}\left(\frac{3}{2} - \frac{1}{5} - \frac{1}{6}\right)$$

$$\Rightarrow \frac{1}{2}\sum_{k=2}^{n} \frac{1}{(k+1)(k-1)} = S_n = \frac{1}{2}\left(\frac{3}{2} - \frac{1}{n} - \frac{1}{n+1}\right)$$

Notice that, due to cancellations, the partial sum has only a small number of terms. Now we may easily take a limit as follows:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{2}\left(\frac{3}{2} - \frac{1}{n} - \frac{1}{n+1}\right) = \frac{1}{2}\left[\frac{3}{2} - 0 - 0\right] = \frac{3}{4}$$

$$\Rightarrow \boxed{\sum_{k=2}^{\infty} \frac{1}{k^2-1} = \lim_{n\to\infty} S_n = \frac{3}{4}}$$

Ex: 2.) Find the sum of the sequence.

$$\sum_{k=2}^{\infty} -\frac{7}{10^{k-1}}$$

Solution:

$$\sum_{k=2}^{\infty} \frac{7}{10^{k-1}} = \sum_{k=1}^{\infty} \frac{7}{10^{k}} = \sum_{k=0}^{\infty} 7\left(\frac{1}{10}\right)^{k+1} = \sum_{k=0}^{\infty} \frac{7}{10}\left(\frac{1}{10}\right)^{k}$$

This geometric series is convergent since $\frac{1}{10} < 1$ and $a = \frac{7}{10}$. This implies that the series converges as follows:

$$\boxed{\sum_{k=0}^{\infty} \frac{7}{10}\left(\frac{1}{10}\right)^{k} = \frac{\frac{7}{10}}{1-\frac{1}{10}} = \frac{7}{9}}$$

1. Does the series $\displaystyle\sum_{k=1}^{10^{50,000^{200}}} (4,508,624,135,024)^k$ converge? Why or why not?

Since it's a finite sum series, then it converges.

- **Directions**: Find the sum of the following sequence if they exist or state that they do not converge and why.

2. $\displaystyle\sum_{k=1}^{\infty} 1^k$

$\boxed{\text{diverges}}$ because it's a geometric series with $|r| = 1$.

3. $\displaystyle\sum_{k=0}^{\infty} 5\left(\frac{4}{5}\right)^k \Rightarrow = 5\sum_{k=0}^{\infty}\left(\frac{4}{5}\right)^k = 5\,\frac{1}{1-\frac{4}{5}} = \boxed{25}$ converges because

$|r| = |\frac{4}{5}| = \frac{4}{5} < 1$ converges by definition of geometric series.

4. $\displaystyle\sum_{k=4}^{\infty} 8\left(\frac{-2}{3}\right)^{k-3} \Rightarrow = 8\sum_{k=1}^{\infty}\left(\frac{-2}{3}\right)^k = 8\cdot\left(\frac{-2}{3}\right)\sum_{k=0}^{\infty}\left(\frac{-2}{3}\right)^k$

$|r| = |\frac{-2}{3}| = \frac{2}{3} < 1$ converges. $\displaystyle = \frac{-16}{3}\cdot\frac{1}{1+\frac{2}{3}} = \boxed{\frac{-16}{5}}$

5. $\sum_{k=2}^{\infty} \left(\frac{\pi}{3.14}\right)^k$

$$\frac{\pi}{3.14} \approx 1.000507215 > 1$$

$$\frac{\pi}{3.14} > 1 \Rightarrow \pi > 3.14 \quad \text{diverges}$$

OR $\quad |r| = \left|\frac{\pi}{3.14}\right| > 1$ diverges

6. $\sum_{k=400}^{\infty} \left(\frac{3.14}{\pi}\right)^{k-399}$

$$\Rightarrow = \sum_{k=1}^{\infty} \left(\frac{3.14}{\pi}\right)^k = \frac{3.14}{\pi} \sum_{k=0}^{\infty} \left(\frac{3.14}{\pi}\right)^k$$

$$= \frac{3.14}{\pi} \cdot \frac{1}{1 - \frac{3.14}{\pi}} \approx \boxed{1971.55}$$

7. $\sum_{k=3}^{\infty} \frac{4}{k^2-4}$

$$\sum_{k=3}^{\infty} \frac{4}{k^2-4} = \sum_{k=3}^{\infty} \frac{4}{(k-2)(k+2)} = \lim_{n\to\infty} \boxed{\sum_{k=3}^{n} \frac{4}{(k-2)(k+2)}}$$

Using partial fractions (Cover Method)

$$\frac{4}{(k-2)(k+2)} = \frac{A}{k-2} + \frac{B}{k+2} \Rightarrow \text{So, } \boxed{\frac{4}{(k-2)(k+2)} = \frac{1}{k-2} - \frac{1}{k+2}}$$

$\boxed{k=2}$ $\quad \boxed{k=-2}$

$$A = \frac{4}{2+2} = \frac{4}{4} = 1 \Rightarrow \boxed{A=1}$$

$$B = \frac{4}{-2-2} = \frac{4}{-4} = -1 \Rightarrow \boxed{B=-1}$$

Thus, $\sum_{k=3}^{n} \frac{4}{(k-2)(k+2)} = \sum_{k=3}^{n} \left[\frac{1}{k-2} - \frac{1}{k+2}\right] = \left[1 - \frac{1}{5}\right] + \left[\frac{1}{2} - \frac{1}{6}\right] +$

$$\left[\frac{1}{3} - \frac{1}{7}\right] + \left[\frac{1}{4} - \frac{1}{8}\right] + \cdots + \left[\frac{1}{n-3} - \frac{1}{n+1}\right] + \left[\frac{1}{n-2} - \frac{1}{n+2}\right]$$

$$\Rightarrow \lim_{n\to\infty} \left[\cancelto{0}{\frac{1}{n-2}} - \cancelto{0}{\frac{1}{n+2}}\right] = \underline{0} \quad \text{convergent.}$$