Mohammed Kaabar — Improper Integrals — October 6^{th}, 2015

* Improper Integrals:

Definition: Improper Integral of Type I

called "improper" contains $-\infty$ and ∞

ⓐ If $\int_a^t f(x)\,dx$ exists for every $t \geq a$, then

$$\boxed{\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx}$$

provided the limit exists (as a finite number).

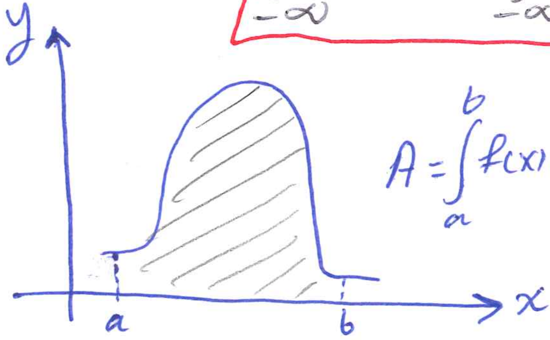
ⓑ If $\int_t^b f(x)\,dx$ exists for any $t \leq b$, then

$$\boxed{\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx}.$$

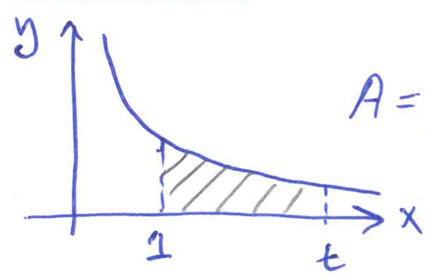
* The improper integrals $\boxed{\int_a^\infty f(x)\,dx}$ and $\boxed{\int_{-\infty}^b f(x)\,dx}$ are called **convergent** if limit exists and is a finite number. Otherwise, It's **divergent** if limit doesn't exist or is infinite.

ⓒ If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are **convergent**, then we define

$$\boxed{\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx}.$$



$A = \int_a^b f(x)\,dx$



$A = \int_1^\infty f(x)\,dx \Rightarrow \lim_{t\to\infty}\int_1^t f(x)\,dx$

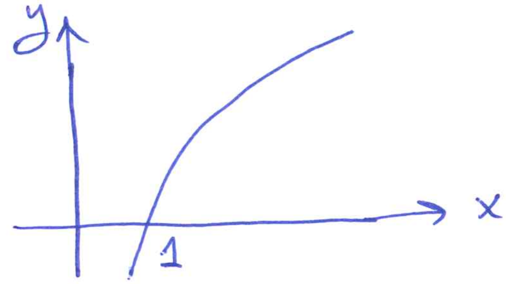
Example 1: Evaluate $\int_1^\infty \frac{1}{x}\,dx$.

Solution:

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \boxed{\int_1^t \frac{1}{x}\,dx} = \lim_{t\to\infty}\left[\ln(t) - \cancel{\ln(1)}^{\,0}\right] = \lim_{t\to\infty} \ln(t)$$

$$\ln(x)\Big|_1^t$$

$= +\infty$ divergent



Example 2: Evaluate $\int_2^\infty \frac{1}{x^2}\,dx$.

Solution:

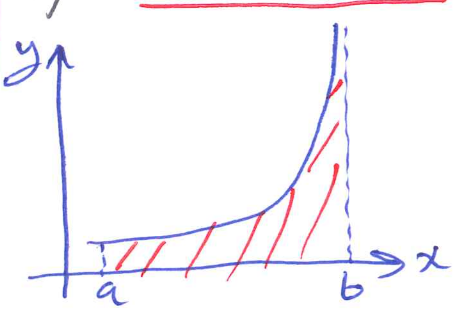
$$\int_2^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty} \boxed{\int_2^t \frac{1}{x^2}\,dx} = \lim_{t\to\infty}\left[-\cancel{\frac{1}{t}}^{\,0} + \frac{1}{2}\right] = \frac{1}{2} \quad \text{convergent.}$$

$$\int_2^t x^{-2}\,dx = \frac{x^{-1}}{-1}\Big|_2^t = -\frac{1}{x}\Big|_2^t$$

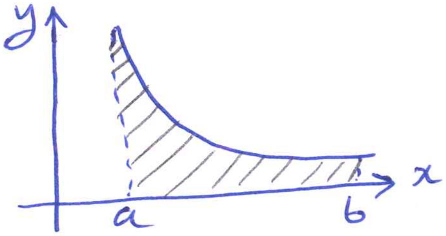
* **Theorem:** $\int_1^\infty \frac{1}{x^p}\,dx$ is convergent if $\boxed{p>1}$, and it's divergent if $\boxed{p\le 1}$.

* **Definition of Improper Integrals (Type II)**

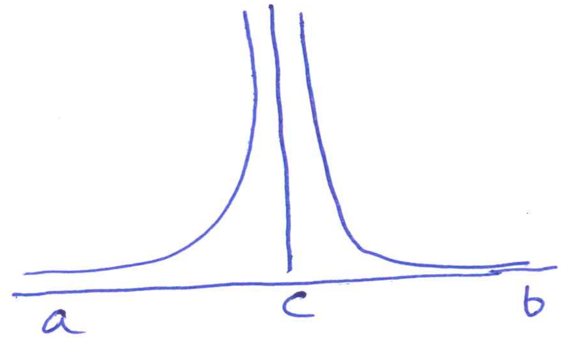
ⓐ If f is continuous on $[a,b)$ and is discontinuous at b, then $\boxed{\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx}$ if limit exists (as a finite number)



(b) If f is continuous on $(a, b]$ and is discontinuous at a, then $\boxed{\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx}$ if this limit exists as a finite number.



(c) If f is discontinuous at c, where $\boxed{a < c < b}$ and both $\boxed{\int_a^c f(x)\,dx}$ and $\boxed{\int_c^b f(x)\,dx}$ convergent, then we define $\boxed{\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx}$.



Example 3: Evaluate $\int_0^\infty \frac{1}{(x-1)^2}\,dx$.

Solution:

$\int \frac{1}{(x-1)^2}\,dx$

$u = x-1 \quad \int \frac{1}{u^2}\,du =$

$du = dx \quad \int u^{-2}\,du = \frac{u^{-1}}{-1} + c$

$= \boxed{\frac{-1}{(x-1)} + c}$

$$\int_0^\infty \frac{1}{(x-1)^2}\,dx = \int_0^1 \frac{1}{(x-1)^2}\,dx + \int_1^\infty \frac{1}{(x-1)^2}\,dx$$

$$= \int_0^1 \frac{1}{(x-1)^2}\,dx + \int_1^2 \frac{1}{(x-1)^2}\,dx + \int_2^\infty \frac{1}{(x-1)^2}\,dx$$

$$= \lim_{t \to 1^-} \int_0^t \frac{1}{(x-1)^2}\,dx + \lim_{t \to 1^+} \int_t^2 \frac{1}{(x-1)^2}\,dx + \lim_{t \to \infty} \int_2^t \frac{1}{(x-1)^2}\,dx$$

$$= \lim_{t \to 1^-} \Big[\underbrace{\frac{-1}{t-1}}_{-\infty} + \frac{1}{-1}\Big] + \lim_{t \to 1^+} \Big[-1 - \underbrace{\frac{1}{t-1}}_{-\infty}\Big] + \lim_{t \to \infty} \Big[\overbrace{\frac{-1}{t-1}}^{0} + 1\Big]$$

Divergent Divergent Convergent

$\Longrightarrow$ So, it's $\boxed{\text{divergent}}$ ☒
